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A one-dimensional model of viscous blood flow in an elastic vessel

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ABSTRACT

In this paper we present a one-dimensional model of blood flow in a vessel segment with an elastic wall consisting of several anisotropic layers. The model involves two variables: the radial displacement of the vessel's wall and the pressure, and consists of two coupled equations of parabolic and hyperbolic type. Numerical simulations on a straight segment of a blood vessel demonstrate that the model can produce realistic flow fields that may appear under normal conditions in healthy blood vessels; as well as flow that could appear during abnormal conditions. In particular we show that weakening of the elastic properties of the wall may provoke a reverse blood flow in the vessel.

1. Introduction

Blood vessels form one of the most complicated and important systems in the human body. The system is exposed to various risks and is poorly amenable to medical treatments. A wide variety of diseases and ailments are caused by disturbances in the blood circulatory system. This makes understanding the system important; and mathematical models of different levels of complexity useful.

The existing one-dimensional models are based on three relations accepted a priori: conservation of fluid mass and momentum together with a linear stationary tube law connecting the cross-section of the tube with the pressure. As a result, after an intrinsic linearization, one obtains a hyperbolic type equation with respect to pressure; or another longitudinal variable. Such models are described in details in [4,13,22,24,25], where one can find a more detailed analysis and references to miscellaneous variants of such models.

In contrast to the existing one-dimensional models, see [26], our model, rigorously derived in [7–10] by means of asymptotic analysis and a dimension reduction procedure, leads to a *hyperbolic-parabolic* type system of differential equations. The model includes an analysis of the kinematic interaction between the elastic walls and the blood flow, and, e.g., makes the above-mentioned traditional tube law non-local and non-stationary.

In our work we focus on the blood flow in a single artery. The goals are twofold. First, to present a simple (one-dimensional, linear) model of a straight segment of an artery, which takes into account the elastic properties of vessels wall, but neglect the surrounding muscle tissue, cf. [7,8]. This model is based on modern techniques from asymptotic analysis and contain as a particular case the standard one dimensional model. Second, to find physical parameters of the system which can provide a

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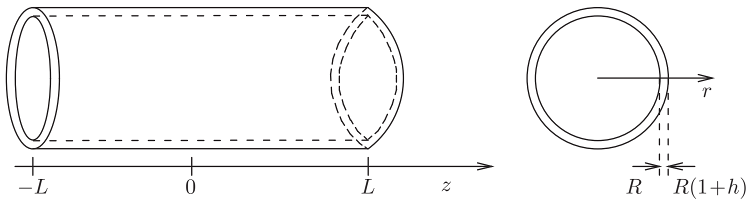

Fig. 1. The geometry of the blood vessel. The thickness of the vessel wall h is small compared to the radius R and the length $2L$.

reverse blood flow (in the direction to the heart). A reverse flow contradicts the purpose of the artery and does not happen during normal conditions. A reverse flow can be caused, for example, by weakening the elastic properties of the vessel due to aging of the collagen fibers or dissection of the wall. The point is that the system of differential equation serving as the mathematical model for the blood flow in the vessel segment under consideration represents a coupling of a hyperbolic and a parabolic equation. These equations work in discord, and therefore the model can reproduce a wide range of flow regimes. If, e.g, a disease provokes a mismatch in the physical parameters the system can be brought out of the normal working regime and mimic flow behavior that occur for patients under influence of cardiovascular diseases. In particular, a reversal of the blood flow in the aorta, can happen in patients suffering from cardiac dysrhythmia, or arrhythmia.

This paper is organized as follows: in Section 2 we present our mathematical model, for details of a derivation of this model we refer to [7–10]. In Section 3 we describe the numerical implementation, and also present results from two test simulations that demonstrate that our model reproduces realistic flow patterns. Finally in Section 4 we give some conclusions and discuss the future development of the model.

2. A mathematical model of blood flow

In this section we develop a mathematical model that describes a single cylindrical shaped blood vessel with a circular cross-section. We try to reduce the complexity of the model to the necessary minimum, that still supports new effects, and take into account only the principal terms in the asymptotic expansions of the physical fields. Although the dimension reduction procedure used also gives us expressions for the higher order terms, which can lead to models of increased accuracy, our goal of finding physical conditions where a reverse flow appears is achived within the simple model. Let

$$\Theta_h^R = \{x = (y, z) \,:\, r = |y| < R(1+h),\ z \in (-L, L)\} \tag{1}$$

be a circular cylinder consisting of a thin elastic wall,

$$\Sigma_h^R = \Theta_h^R \setminus \Theta_0^R, \tag{2}$$

and the interior of the vessel[1] Θ_0^R. The dimensionless parameters h and $\delta = R/L$ are supposed to be small and $h << \delta$. The surface where the blood flow interacts with the wall of the vessel is denoted by

$$\Gamma_0^R = \{x \,:\, r = R, z \in (-L, L)\}. \tag{3}$$

The domain is illustrated in Fig. 1.

The blood flow is assumed to be symmetric around the z–axis. The physical fields we are interested in are the velocity vector $\mathbf{v} = (v_r, v_z)^T$ for the blood flow inside the vessel, the displacement vector for the vessel's wall $\mathbf{u} = (u_r, u_z)^T$, and the pressure p of the blood. Within the Euler approach and under the conditions of small deformations in the wall as well as a relatively slow flow of blood, see Section 2.1, we reduce all constitutive relations to the reference surface (5), where R denotes a constant typical radius, while the actual radius of the vessel clearance is $R + u_r(z, t)$.

The velocity and the pressure satisfy the Navier–Stokes system:

$$\begin{cases} \gamma_b(\partial_t v_r + v_r\partial_r v_r + v_z\partial_z v_r) = \mu(\Delta v_r - r^{-2}v_r) - \partial_r p, \\ \gamma_b(\partial_t v_z + v_r\partial_r v_z + v_z\partial_z v_z) = \mu\Delta v_z - \partial_z p, \end{cases} \tag{4}$$

and the continuity equation,

$$\partial_r v_r + r^{-1}v_r + \partial_z v_z = 0, \tag{5}$$

where we used a cylindrical coordinate system, $\Delta = \partial_r^2 + r^{-1}\partial_r + \partial_z^2$ is the Laplace operator, γ_b is the density of the blood, and μ is the dynamical viscosity coefficient of the blood. The dynamical non-slipping condition:

$$\mathbf{v} = \partial_t \mathbf{u}, \quad \text{on } \Gamma_0^R, \tag{6}$$

where $\partial_t \mathbf{u}$ represents the velocity of the wall's displacement, is imposed on the surface where the blood flow interacts with the vessel's wall.

[1] In [10] it was proved that a circular form of the cross-section is optimal, see also [9].