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A new numerical technique for solving the local fractional diffusion equation: Two-dimensional extended differential transform approach

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ABSTRACT

In this article, we first propose a new numerical technique based upon a certain two-dimensional extended differential transform via local fractional derivatives and derive its associated basic theorems and properties. One example of testing the local fractional diffusion equation is then considered. The numerical result presented here illustrates the efficiency and accuracy of the proposed computational technique in order to solve the partial differential equations involving local fractional derivatives.

1. Introduction

Fractional derivative operators (FDOs) have successfully been observed to provide mathematical analytic tools with potential for applications in mathematical sciences, physics and mechanics (see [1,2]). Fractional partial differential equations (FDEs) describing anomalous phenomena in real world have been witnessed over the past 50 years (see [3–5]). There are some methods for finding analytical, numerical and numerical-analytical solutions to FDEs. Several analytical techniques, such as the Adomian decomposition [6], variational iteration [7] and homotopy analysis [8] methods, were successfully investigated for solving various families of FDEs. Furthermore, several approaches for finding numerical solutions for FDEs, such as the Kansa method [9], discontinuous spectral element [10] and Chebyshev spectral methods [11], and other methods [12–14] were also discussed. A semi-numerical technique for solving FDEs, called differential transform (DT) (see [15,16]), was proposed and developed to deal with modified KdV [17], oscillator [18], and coupled Burgers' [19] equations with FDOs.

The theory of the local FDOs [20–22] was adopted to describe non-differentiable problems from fractal physical phenomena, which are not dealt with the above non-local operator [1–5]. Hence, several analytical and numerical approaches for local fractional partial differential equations (LFPDEs) [23–28] were formulated. In [23], the one-dimensional extended DT (ODEDT) was proposed. However, the two-dimensional extended differential transform (TDEDT) is not discussed. The main target of this article is to point out a semi-numerical technique, called the TDEDT, in order to find the non-differentiable solution for the local fractional diffusion equation (LFDE). The layout of this article is presented as follows. In Section 2, the notations and conceptions

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of LFDOs are given. In Section 3, the TDEDT via LFDO and its associated basic theorems and properties are discussed. In Section 4, the non-differentiable solution to the LFDE is investigated. Finally, the conclusions are presented in Section 5.

2. Preliminaries

Let $C_\kappa(\xi,\zeta)$ be a set of the non-differentiable functions with the fractal dimension $\kappa(\kappa \in (0,1])$ (see [20,28]).

For $\Phi(\lambda) \in C_\kappa(\xi,\zeta)$, the LFDO of $\Phi(\lambda)$ of order $\kappa(\kappa \in (0,1])$ at the point $\lambda = \lambda_0$ is defined as follows (see [20,23–30]):

$$D^{(\kappa)}\Phi(\lambda_0) = \frac{d^\kappa \Phi(\lambda_0)}{d\lambda^\kappa} = \lim_{\lambda \to \lambda_0} \frac{\Delta^\kappa(\Phi(\lambda) - \Phi(\lambda_0))}{(\lambda - \lambda_0)^\kappa}, \tag{1}$$

where

$$\Delta^\kappa(\Phi(\lambda) - \Phi(\lambda_0)) \cong \Gamma(1+\kappa)[\Phi(\lambda) - \Phi(\lambda_0)]. \tag{2}$$

The LFPDE of $\Phi(\lambda,\omega)$ of order κ $(\kappa \in (0,1])$ with respect to λ at the point (λ_0,ω) is defined as follows (see [20]):

$$\frac{\partial^\kappa \Phi(\lambda,\omega)}{\partial \lambda^\kappa} = \lim_{\lambda \to \lambda_0} \frac{\Delta^\kappa(\Phi(\lambda,\omega) - \Phi(\lambda_0,\omega))}{(\lambda - \lambda_0)^\kappa}, \tag{3}$$

where

$$\Delta^\kappa(\Phi(\lambda,\omega) - \Phi(\lambda_0,\omega)) \cong \Gamma(1+\kappa)[\Phi(\lambda,\omega) - \Phi(\lambda_0,\omega)]. \tag{4}$$

Similarly, the LFPDE of $\Phi(\lambda,\omega)$ of order $\kappa(\kappa \in (0,1])$ with respect to ω at the point (λ,ω_0) is defined as follows (see [20]):

$$\frac{\partial^\kappa \Phi(\lambda,\omega)}{\partial \omega^\kappa} = \lim_{\omega \to \omega_0} \frac{\Delta^\kappa(\Phi(\lambda,\omega) - \Phi(\lambda,\omega_0))}{(\omega - \omega_0)^\kappa}, \tag{5}$$

where

$$\Delta^\kappa(\Phi(\lambda,\omega) - \Phi(\lambda,\omega_0)) \cong \Gamma(1+\kappa)[\Phi(\lambda,\omega) - \Phi(\lambda,\omega_0)]. \tag{6}$$

In view of (1), the LFPD of $\Theta(l,m)$ of order $m\kappa(\kappa \in (0,1])$ is given by (see [20,23–28])

$$\frac{\partial^{m\kappa} \Phi(\lambda,\omega)}{\partial \lambda^{m\kappa}} = \overbrace{\frac{\partial^\kappa}{\partial \lambda^\kappa} \cdots \frac{\partial^\kappa}{\partial \lambda^\kappa}}^{m \text{ times}} \Phi(\lambda,\omega). \tag{7}$$

3. The TDEDT technique via LFDO

In this section, the conceptions and theorems of the TDEDT via LFDO are presented.

Definition 1. If $\theta(\tau,\upsilon)$ is a local fractional analytic function [20] in the domain Ξ, then the TDEDT of the function $\theta(\tau,\upsilon)$ via LFDO is defined as follows:

$$\Theta(\mu,\eta) = \frac{1}{\Gamma(1+\mu\kappa)} \frac{1}{\Gamma(1+\eta\kappa)} \left. \frac{\partial^{(\mu+\eta)\kappa}\theta(\tau,\upsilon)}{\partial \tau^{\mu\kappa}\partial \upsilon^{\eta\kappa}} \right|_{\tau=\tau_0,\upsilon=\upsilon_0}, \quad \tau, \tau_0, \upsilon, \upsilon_0 \in \Xi, \tag{8}$$

where $\mu, \eta = 0, 1, \ldots, n$, $\kappa \in (0,1]$, and the two-dimensional differential inverse transform (TDDIT) of $\theta(\tau,\upsilon)$ in the domain Ξ via LFDO is defined by

$$\theta(\tau,\upsilon) = \sum_{\mu=0}^{\infty}\sum_{\eta=0}^{\infty} \Theta(\mu,\eta)(\tau-\tau_0)^{\mu\kappa}(\upsilon-\upsilon_0)^{\eta\kappa}, \quad \mu,\eta \in \Sigma. \tag{9}$$

Here $\Theta(\mu,\eta)$ is called the non-differentiable spectrum of $\theta(\tau,\upsilon)$ in the domain Σ.

In fact, from the expressions (8) and (9), we have

$$\theta(\tau,\upsilon) = \sum_{\mu=0}^{\infty}\sum_{\eta=0}^{\infty} \frac{(\tau-\tau_0)^{\mu\kappa}}{\Gamma(1+\mu\kappa)} \frac{(\upsilon-\upsilon_0)^{\eta\kappa}}{\Gamma(1+\eta\kappa)} \left. \frac{\partial^{(\mu+\eta)\kappa}\theta(\tau,\upsilon)}{\partial \tau^{\mu\kappa}\partial \upsilon^{\eta\kappa}} \right|_{\tau=\tau_0,\upsilon=\upsilon_0}, \tag{10}$$

which is in agreement with the two-dimensional local fractional Taylor series expansion of the function $\theta(\tau,\upsilon)$ [20].

For $\tau_0 = 0$ and $\upsilon_0 = 0$, from (8) and (9) we obtain the TDEDT via LFDO [25]:

$$\Theta(\mu,\eta) = \frac{1}{\Gamma(1+\mu\kappa)} \frac{1}{\Gamma(1+\eta\kappa)} \left. \frac{\partial^{(\mu+\eta)\kappa}\theta(\tau,\upsilon)}{\partial \tau^{\mu\kappa}\partial \upsilon^{\eta\kappa}} \right|_{\tau=0,\upsilon=0}, \quad \tau,\upsilon \in \Xi \tag{11}$$

and the TDIDT via LFDO [25]: