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# Numerical solving nonlinear integro-parabolic equations by the monotone weighted average method

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ABSTRACT

The paper deals with numerical solving nonlinear integro-parabolic problems of Volterra type based on the weighted average method. A monotone iterative method is presented. Construction of initial upper and lower solutions is given. Existence and uniqueness of a solution to the nonlinear difference scheme are established. An analysis of convergence rates of the monotone iterative method is given. Numerical experiments are presented.



## 1. Introduction

Integro-parabolic differential equations of Volterra type arise in the chemical, physical and engineering sciences (see [8] for details). In this paper we give a numerical treatment for nonlinear integro-parabolic differential equations of Volterra type. The parabolic problem under consideration is given in the form

$$\begin{aligned} &u_t - Lu + f(x,t,u) + \int_0^t g_0(x,t,s,u(x,s))ds = 0, \quad (x,t) \in \omega \times (0,T], \\ &u(x,t) = h(x,t), \quad (x,t) \in \partial\omega \times (0,T], \\ &u(x,0) = \psi(x), \quad x \in \overline{\omega}, \end{aligned} \tag{1}$$

where $\omega$ is a connected bounded domain in $\mathbb{R}^\kappa$ ($\kappa = 1, 2, \ldots$) with boundary $\partial\omega$. The linear differential operator $L$ is given by

$$Lu = \sum_{\alpha=1}^{\kappa} \frac{\partial}{\partial x_\alpha}\left(D(x,t)\frac{\partial u}{\partial x_\alpha}\right) + \sum_{\alpha=1}^{\kappa} v_\alpha(x,t)\frac{\partial u}{\partial x_\alpha},$$

where the coefficients of the differential operators are smooth and $D$ is positive in $\overline{\omega} \times [0,T]$. It is also assumed that the functions $f, g_0, h$ and $\psi$ are smooth in their respective domains.

For approximating the semilinear problem (1), we shall use the weighted average scheme or the $\theta$-method [6]. This nonlinear 10-point difference scheme can be regarded as taking a weighted average of the explicit and implicit schemes. In order to practically compute a solution of the nonlinear weighted average scheme, one requires an efficient numerical method. A fruitful method for solving nonlinear difference schemes is the method of upper and lower solutions and associated monotone iterates. By using upper and lower solutions as two initial iterations, one can construct two monotone sequences which converge monotonically from above and below, respectively, to a solution of the problem.

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Monotone iterative schemes for solving nonlinear parabolic equations are used in [1,2,7,9,10,12,15]. In [11], a monotone iterative method for solving nonlinear integro-parabolic equations of Fredholm type is presented. Here, the two important points in investigating the monotone iterative method concerning a stopping criterion on each time level and estimates of convergence rates, in the case of solving linear discrete systems on each time level inexactly, were not given.

In [3], we constructed and investigated the monotone iterative method, based on the fully implicit approximation of the differential operator in the case when on each time level nonlinear difference schemes were solved inexactly, and gave an analysis of convergence rates of the monotone iterative method. Since the fully implicit approximation possesses only the first order accuracy in time, in [3], the integral term is discretized by the Riemann sum (the rectangular rule). In this paper, we extend the approach from [3] on the weighted average difference scheme. Taking into account that the Crank–Nicolson approximation of the differential operator has the second order accuracy in time [4], in this paper, we approximate the integral term by using the trapezoidal rule. In [3], it is assumed that on each time level, a pair of initial upper and lower initial solutions is given in advance, we now construct initial upper and lower solution explicitly.

The structure of the paper as follows. In Section 2, we introduce a nonlinear difference scheme for the numerical solution of (1), based on the $\theta$-method. A monotone iterative method is presented in Section 3. Existence and uniqueness of the solution to the nonlinear difference scheme are established. An analysis of convergence rates of the monotone iterative method is given. Convergence of the nonlinear difference scheme to the nonlinear integro-parabolic problem (1) is established. Section 4 presents results of numerical experiments.

## 2. The nonlinear difference scheme

On the domains $\overline{\omega}$ and $[0, T]$, we introduce meshes $\overline{\omega}^h$ and $\overline{\omega}^\tau$, respectively. For solving (1), consider the nonlinear two-level weighted average difference scheme

$$\begin{aligned}&\tau_k^{-1}[U(p,t_k)-U(p,t_{k-1})]+\theta\mathcal{L}^hU(p,t_k)+(1-\theta)\mathcal{L}^hU(p,t_{k-1})+\\&\theta[f(p,t_k,U)+g(p,t_k,U)]+(1-\theta)[f(p,t_{k-1},U)+g(p,t_{k-1},U)]=0,\\&(p,t_k)\in\omega^h\times(\overline{\omega}^\tau\setminus\{0\}),\end{aligned}\tag{2}$$

with the boundary and initial conditions

$$\begin{aligned}&U(p,t_k)=h(p,t_k),\quad (p,t_k)\in\partial\omega^h\times(\overline{\omega}^\tau\setminus\{0\}),\\&U(p,0)=\psi(p),\quad p\in\overline{\omega}^h,\end{aligned}$$

where $\theta=\text{const}$, $0<\theta\le 1$, $\partial\omega^h$ is the boundary of $\overline{\omega}^h$ and time steps $\tau_k=t_k-t_{k-1}$, $k\ge 1$, $t_0=0$. When no confusion arises, we write $f(p,t_k,U(p,t_k))=f(p,t_k,U)$.

This difference scheme can be regarded as taking a weighted average of the explicit scheme ($\theta=0$) and the fully implicit scheme ($\theta=1$).

The difference operator $\mathcal{L}^h$ is defined by

$$\mathcal{L}^hU(p,t_k)=d(p,t_k)U(p,t_k)-\sum_{p'\in\sigma'(p)}a(p',t_k)U(p',t_k),\tag{3}$$

where $\sigma'(p)=\sigma(p)\setminus\{p\}$, $\sigma(p)$ is a stencil of the scheme at an interior mesh point $p\in\omega^h$. We make the following assumptions on the coefficients of the difference operator $\mathcal{L}^h$:

$$\begin{aligned}&d(p,t_k)>0,\quad a(p',t_k)\ge 0,\quad p'\in\sigma'(p),\\&d(p,t_k)-\sum_{p'\in\sigma'(p)}a(p',t_k)\ge 0,\quad (p,t_k)\in\omega^h\times(\overline{\omega}^\tau\setminus\{0\}).\end{aligned}\tag{4}$$

The integral in (1) is approximated by finite sums in (2), based on the trapezoidal rule,

$$g(p,t_{k-1},U)=\sum_{l=1}^{k-1}\frac{\tau_l}{2}[g_0(p,t_{k-1},t_{l-1},U(p,t_{l-1}))+g_0(p,t_{k-1},t_l,U(p,t_l))],$$

$$g(p,t_k,U)=\sum_{l=1}^{k}\frac{\tau_l}{2}[g_0(p,t_k,t_{l-1},U(p,t_{l-1}))+g_0(p,t_k,t_lU(p,t_l))].$$

We also assume that the mesh $\overline{\omega}^h$ is connected. It means that for two interior mesh points $\tilde{p}$ and $\hat{p}$, there exists a finite set of interior mesh points $\{p_1,p_2,\dots,p_s\}$ such that

$$p_1\in\sigma'(\tilde{p}),\quad p_2\in\sigma'(p_1),\dots,\quad p_s\in\sigma'(p_{s-1}),\quad \hat{p}\in\sigma'(p_s).\tag{5}$$

On each time level $t_k$, $k\ge 1$, introduce the linear difference problem

$$\begin{aligned}&\big(\theta\mathcal{L}^h(p,t_k)+\tau_k^{-1}+\theta c(p,t_k)\big)W(p,t_k)=\Phi(p,t_k),\quad p\in\omega^h,\\&c(p,t_k)\ge 0,\quad W(p,t_k)=h(p,t_k),\quad p\in\partial\omega^h.\end{aligned}\tag{6}$$