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A new algorithm for nonlinear fourth order multi-point boundary value problems

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ABSTRACT

In this paper, a new algorithm is presented to solve the nonlinear fourth-order differential equations with complicated boundary conditions. The approach combines the Quasi-Newton's method and the simplified reproducing kernel method. It is worth mentioning that the Quasi-Newton's method is proposed for solving the nonlinear differential equations for the first time. Meanwhile, the simplified reproducing kernel method is applied to solve the linear equations which are obtained from the conversion of the Quasi-Newton's method, avoiding the time-consuming Schmidt orthogonalization process. Moreover, the reproducing kernel space and its reproducing kernel are reasonably simple as no considering of the complicated boundary conditions. Furthermore the present scheme is employed successfully on some numerical examples.

1. Introduction

The nonlinear equations arise in various fields of applied mathematics and physics. Lots of physical problems can be described by nonlinear differential equations. With the development of science and technology, complicated boundary value problems (BVPs for short) have been put forward, which make the problems more practical and their solutions more compatible. The investigations have been shown in the papers [1–7] on the existence and multiplicity of solutions for the fourth-order four-point BVPs. Meanwhile the uniqueness of solutions has also been determined. So far, the numerical algorithm of these problems has attracted considerable attention. Many new numerical algorithms have been proposed as follows. In the article [8], the author provided a method for a class of second-order three-point BVPs by converting the original problem into an equivalent integro-differential equation. Another was an upper and lower solution method for fourth-order four-point BVPs [9]. Geng and Cui [10] proposed a new reproducing kernel method for linear nonlocal boundary value problem, while Dehghan and Tatari [11] supplied an algorithm for solving multi-point BVPs by the Adomian decomposition method. Most recently, Li and Wu [12] presented homotopy perturbation method and reproducing kernel method to solve singular fourth-order four-point BVPs. Duan and R.Rach [14] proposed a new modification of the Adomian decomposition method for solving BVPs for higher order nonlinear differential equations. Chen [13] put forward perturbation method for nonlocal impulsive evolution equations.

In the present work, a novel algorithm, that is, the combination of the Quasi-Newton's method and the simplified reproducing kernel method, is proposed to solve the nonlinear differential equation of the following form

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$$\begin{cases} u''''(x) + a_3(x)u'''(x) + a_2(x)u''(x) + a_1(x)u'(x) + \mathcal{N}(u) = f(x), x \in [a,b], \\ u(a) = \alpha_1,\ u'''(c) = \alpha_2,\ u''(d) = \alpha_3,\ u(b) = \alpha_4. \end{cases} \tag{1}$$

where $f(x), a_i(x) \in C[a,b], i = 1,2,3, \mathcal{N}$ is nonlinear.

The outline of this paper is as follows: In Section 2, the definition of Fréchet derivative is recalled. Based on Fréchet derivative, the Quasi-Newtons method is proposed to solve the problem (1). In Section 3, the simplified reproducing kernel method is presented to solve the linear equations obtained by the Quasi-Newton's method. In Section 4, the presented algorithms are applied to some numerical experiments. Then we end with some conclusions in Section 5.

2. The Quasi-Newton's method

2.1. Fréchet derivative

In mathematics, the *Fréchet derivative*, proposed by D. Behmardi [15], is a derivative defined on Banach spaces. It is often used to formalize the functional derivative commonly used in physics, particularly in Quantum field theory.

Definition 2.1. Let $\mathcal{F} : X \to Y$, where X and Y are Banach spaces. Then a bounded and linear operator $\mathcal{A} : X \to Y$ is called a *Fréchet derivative* of $\mathcal{F}$ at $u \in X$ if

$$\lim_{h\to 0} \frac{\| \mathcal{F}(u+h) - \mathcal{F}(u) - \mathcal{A}(h) \|}{\|h\|} = 0$$

for all $h \in X$, denoted by $\mathcal{F}'(u)$.

The following lemma can be obtained easily by the above definition, without proof.

Lemma 2.2. *Let $\mathcal{F}$ be a linear operator of the Banach space X into the Banach space Y. Then $\mathcal{F}' = \mathcal{F}$.*

Now, we introduce two examples for the *Fréchet derivative* of the nonlinear operator.

Example 2.1. Let $\mathcal{F}$: $C^2 \to C$, $\mathcal{F}$: $u \longmapsto u'' + u^2$. Then $\mathcal{F}'(u_0)$: $u \longmapsto u'' + 2u_0u$.

Example 2.2. Let $\mathcal{F}$: $C \to C$, $\mathcal{F}$: $u \longmapsto e^{-4u}$. Then $\mathcal{F}'(u_0)$: $u \longmapsto -4e^{-4u_0}u$

2.2. The Quasi-Newton's method

Solving nonlinear equations is one of the most basic problems in numerical analysis. To solve these equations, an iterative method called Newton's method is used extensively. For a nonlinear function $y = f(x)$, at the point $(x_0, f(x_0))$ the tangent equation is

$$y - f(x_0) = f'(x_0)(x - x_0). \tag{2}$$

Therefore, the Newton's iterative scheme for the nonlinear equation $f(x) = 0$ is obtained

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, k = 0, 1 \ldots. \tag{3}$$

In a similar manner, we define the tangent equation of the nonlinear differential operator $\mathcal{F} = \mathcal{F}(u)$ at $(u_0, \mathcal{F}(u_0))$, which can be written as

$$\mathcal{F} - \mathcal{F}(u_0) = \mathcal{F}'(u_0)(u - u_0). \tag{4}$$

Defining an operator $\mathcal{F}$: $C^4[a,b] \to C[a,b]$. By Eq. (1), putting

$$\mathcal{F}(u) \triangleq u'''' + a_3(x)u'''(x) + a_2(x)u''(x) + a_1(x)u'(x) + \mathcal{N}(u). \tag{5}$$

Lemma 2.3. *The Fréchet derivative $\mathcal{F}'(u_0) :\ C^4[a,b] \to C[a,b]$ is*

$$\mathcal{F}'(u_0) : u \longmapsto u'''' + a_3(x)u'''(x) + a_2(x)u''(x) + a_1(x)u'(x) + \mathcal{N}'(u_0)u. \tag{6}$$

Theorem 2.4. *The Quasi-Newton Iteration schemes of the Eq. (1) is presented as*

$$\begin{cases} u''''_{k+1} + a_3(x)u'''_{k+1} + a_2(x)u''_{k+1} + a_1(x)u'_{k+1} + \mathcal{N}'(u_k)(u_{k+1} - u_k) + \mathcal{N}(u_k) = f(x), \\ u_{k+1}(a) = \alpha_1,\ u'''_{k+1}(c) = \alpha_2,\ u''_{k+1}(d) = \alpha_3,\ u_{k+1}(b) = \alpha_4.,\quad k = 0, 1, \ldots. \end{cases} \tag{7}$$

Thus, a linear differential equation is obtained. Applying the technology to the nonlinear Eq. (1), we choose the polynomial function satisfying the boundary conditions as u_0, then the nonlinear equation is translated to the linear one (7). The simplified reproducing kernel method will be extended to solve this linear equation in part 3.