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Some remarks on oscillation of second order neutral differential equations

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ABSTRACT

We study oscillatory behavior of a class of nonlinear second-order neutral differential equations. A new criterion is established that amends related results reported in the literature.

1. Introduction

In this paper, we are concerned with the oscillation of a second-order nonlinear neutral differential equation

$$(r(t)((x(t)+p(t)x(\tau(t)))')^{\alpha})'+q(t)x^{\alpha}(\sigma(t))=0, \tag{1.1}$$

where $t \geq t_0 > 0$, $\alpha \geq 1$ is a quotient of odd positive integers, the functions r, p, q, τ, σ are such that $r, \sigma \in C^1([t_0,\infty),(0,\infty))$, $p, q, \tau \in C([t_0,\infty),\mathbb{R})$. We also suppose that, for all $t \geq t_0$, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\sigma'(t) > 0$, $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\sigma(t) = \infty$, $0 \leq p(t) < 1$, $q(t) \geq 0$, and q does not vanish eventually.

In the sequel, we denote $z(t) := x(t) + p(t)x(\tau(t))$ and assume that solutions x to Eq. (1.1) exist and can be continued indefinitely to the right. As usual, a solution of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros; otherwise, it is termed nonoscillatory. Eq. (1.1) is said to be oscillatory if all its solutions oscillate.

The increasing interest in oscillatory behavior of solutions to second-order neutral differential equations is motivated by their applications in the natural sciences and engineering. We refer the reader to [1–8] and the references cited therein. Therein, Han et al. [2] and Ye and Xu [7] proved several oscillation results for (1.1), some of which we present below for the convenience of the reader. In what follows, we use the notation:

$$\varepsilon := (\alpha/(\alpha+1))^{\alpha+1},\ Q(t) := q(t)(1-p(\sigma(t)))^{\alpha},\ \pi(t) := \int_t^{\infty} r^{-1/\alpha}(s)\mathrm{d}s.$$

Theorem 1.1 (See [7, Theorem 2.3]). *Assume*

$$\pi(t_0) < \infty \tag{1.2}$$

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and let

$$\int^{\infty}\left[Q(t)\pi^{\alpha}(\sigma(t))-\frac{\varepsilon\sigma'(t)}{\pi(\sigma(t))r^{1/\alpha}(\sigma(t))}\right]dt=\infty$$

and

$$\int^{\infty}\left[Q(t)\pi^{\alpha}(t)-\frac{\varepsilon r(\sigma(t))}{\pi(t)(\sigma'(t))^{\alpha}r^{(\alpha+1)/\alpha}(t)}\right]dt=\infty.$$

Then Eq. (1.1) is oscillatory.

Theorem 1.2 (See [2, Theorem 2.1]). *Assume (1.2) and let*

$$p'(t)\geq 0 \quad \text{and} \quad \sigma(t)\leq\tau(t)=t-\tau_0 \quad \text{for} \quad t\geq t_0. \tag{1.3}$$

If there exists a function $\rho\in C^1([t_0,\infty),(0,\infty))$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)Q(s)-\frac{((\rho'(s))_+)^{\alpha+1}r(\sigma(s))}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(s)(\sigma'(s))^{\alpha}}\right]ds=\infty \tag{1.4}$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{q(s)\pi^{\alpha}(s)}{(1+p(s))^{\alpha}}-\frac{\varepsilon}{\pi(s)r^{1/\alpha}(s)}\right]ds=\infty,$$

where $(\rho'(t))_+:=\max\{0,\rho'(t)\}$, then Eq. (1.1) is oscillatory.

On the basis of assumption (1.3), the inaccuracies in Theorem 1.1 have been corrected by Theorem 1.2. In particular, Han et al. [2] pointed out that

$$x(t)\geq(1-p(t))z(t)$$

does not hold eventually when assuming that x is a positive solution of (1.1) and condition (1.2) holds.

The objective of this paper is to improve recent oscillation results due to Han et al. [2] and Ye and Xu [7]. Using the generalized Riccati substitution, a new oscillation criterion for (1.1) is obtained that does not require assumption (1.3).

2. Oscillation results

All functional inequalities considered in this section are assumed to be satisfied for all t sufficiently large. The following auxiliary result collects two useful inequalities that can be found in Jiang and Li [4] and Zhang and Wang [8], respectively.

Lemma 2.1. *Let $\alpha\geq 1$ be a ratio of two odd numbers. Then*

$$A^{(\alpha+1)/\alpha}-(A-B)^{(\alpha+1)/\alpha}\leq\frac{B^{1/\alpha}}{\alpha}[(1+\alpha)A-B], \quad AB\geq 0 \tag{2.1}$$

and

$$-Cv^{(\alpha+1)/\alpha}+Dv\leq\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{D^{\alpha+1}}{C^{\alpha}}, \quad C>0. \tag{2.2}$$

Theorem 2.2. *Assume that (1.2) holds and there exists a function $\rho\in C^1([t_0,\infty),(0,\infty))$ such that (1.4) is satisfied. If there exists a function $\delta\in C^1([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\psi(s)-\frac{\delta(s)r(s)((\varphi(s))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right]ds=\infty, \tag{2.3}$$

where

$$\psi(t):=\delta(t)\left[q(t)\left(1-p(\sigma(t))\frac{\pi(\tau(\sigma(t)))}{\pi(\sigma(t))}\right)^{\alpha}+\frac{1-\alpha}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)}\right],$$

$$p(t)<\pi(t)/\pi(\tau(t)), \quad \varphi(t):=\frac{\delta'(t)}{\delta(t)}+\frac{1+\alpha}{r^{1/\alpha}(t)\pi(t)},$$

and $(\varphi(t))_+:=\max\{0,\varphi(t)\}$, then Eq. (1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1.1) on $[t_0,\infty)$. Without loss of generality, we may assume that there exists a $t_1\geq t_0$ such that $x(t)>0$, $x(\tau(t))>0$, and $x(\sigma(t))>0$ for all $t\geq t_1$. Then $z(t)\geq x(t)>0$, and by virtue of

$$(r(t)(z'(t))^{\alpha})'=-q(t)x^{\alpha}(\sigma(t))\leq 0, \tag{2.4}$$