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Unconditional superconvergence analysis of a new mixed finite element method for nonlinear Sobolev equation

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ABSTRACT

In this paper, a new mixed finite element scheme is proposed for the nonlinear Sobolev equation by employing the finite element pair $Q_{11}/Q_{01} \times Q_{10}$. Based on the combination of interpolation and projection skill as well as the mean-value technique, the τ-independent superclose results of the original variable u in H^1-norm and the variable $\vec{q} = -(a(u)\nabla u_t + b(u)\nabla u)$ in L^2-norm are deduced for the semi-discrete and linearized fully-discrete systems (τ is the temporal partition parameter). What's more, the new interpolated postprocessing operators are put forward and the corresponding global superconvergence results are obtained unconditionally, while previous literature always require certain time step conditions. Finally, some numerical results are provided to confirm our theoretical analysis, and show the efficiency of the method.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex polygonal domain with sufficiently smooth boundary $\partial\Omega$ and $0 < T < \infty$. We develop and analyze a new mixed finite element approximation to the following nonlinear Sobolev equation

$$\begin{cases} u_t - \nabla \cdot (a(u)\nabla u_t + b(u)\nabla u) = f(X,t), & (X,t) \in \Omega \times (0,T], \\ u = 0, & (X,t) \in \partial\Omega \times (0,T], \\ u(X,0) = u_0(X), & X \in \Omega, \end{cases} \tag{1.1}$$

where, $X = (x, y)$, $a(u)$, $b(u)$ and $f(X, t)$ are smooth functions. Assume that there exist constants μ, M, B, such that $0 < \mu \le a(u)$, $b(u) \le M$, $|a'(u) + b'(u)| \le B$.

Sobolev equation arises in the flow of fluids through fissured rock [1], thermodynamics shear in second order fluids, consolidation of clay, and other applications [2]. A lot of simulation methods have been put forward for Sobolev equation. For example, for linear case, Gao and Rui [3] formulated two spliting least-squares mixed finite element procedures, which yielded optimal order error estimates. The expanded mixed finite element methods(FEMs) were elaborated in [4,5], and optimal order error estimates for both the scalar and two vector functions are obtained. As to the nonlinear case, Ewing [6] studied the Galerkin FEM. [7] investigated semi-discrete and fully-discrete schemes and derived an optimal order error estimation of EQ_1^{rot} nonconforming finite element [8,9]. Sun and Yang [10] considered a penalty discontinuous Galerkin conforming FEM.

Furthermore, for many other nonlinear physical systems, the time-dependent optimal error estimates with the linearized Galerkin FEM have been extensively researched, such as nonlinear parabolic problems [11,12], heat and moisture transport system

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in textile materials [13,14], nonlinear Schrödinger equations [15,16], and Navier–Stokes equations [17–19]. Since the boundedness of the numerical solution in L^∞ norm is a key issue in the error analysis, we may use mathematical induction with an inverse inequality to bound the numerical solution

$$\|U_h^n - R_h u^n\|_{L^\infty} \le Ch^{-\frac{d}{2}}\|U_h^n - R_h u^n\|_0 \le Ch^{-\frac{d}{2}}(h^{r+1} + \tau^m), \tag{1.2}$$

where U_h^n and u^n are the finite element solution and the exact solution, respectively, R_h is a certain projection operator. Obviously, the time-step restriction arises from (1.2) immediately, which may result in extremely time-consuming in practical computations. To avoid this difficulty, Li and Sun [20] proposed a linearized backward Euler–Galerkin FEM for a class of nonlinear parabolic PDEs and obtained the unconditional error estimates. The main idea in [20] is the error splitting technique, which splitted the numerical error into two parts, the spatial error and the temporal error. Thus, the estimate of (1.2) will be replaced by

$$\|U_h^n - R_h U^n\|_{L^\infty} \le Ch^{-\frac{d}{2}}\|U_h^n - R_h U^n\|_0 \le Ch^{-\frac{d}{2}}h^{r+1}, \tag{1.3}$$

without any time-restrictions, where U^n is the time-discrete solution. Motivated by this idea, in [21–23], authors investigated the incompressible miscible flow in porous media, the nonlinear Schrödinger equation and miscible displacement in porous media and deduced the unconditional error estimates, respectively.

As we know, for the usual mixed FEM, the two approximating spaces must be chosen carefully so that they can satisfy the so-called Ladyzhenskaya–Babuska–Brezzi (LBB) condition. In order to make this condition to be satisfied easier, Chen and Chen [24] established a mixed variational form for second elliptic problems in which the two approximating spaces only need to fulfill a very simple inclusion relationship. Consequently, based on the nonconforming finite element pair $EQ_1^{rot}/Q_{10} \times Q_{01}$, Shi and Zhang [25] applied the method to a linear Sobolev equation under the semi-discrete and Euler fully-discrete schemes, and obtained the corresponding optimal error estimates and superclose results. Shi and Zhang [26], [27] studied the linear parabolic equation, [26] showed optimal error results of order $O(h)$ by employing the nonconforming finite element pair P_1/P_0, and [27] deduced the superclose results of order $O(h^2)$ as well as the extrapolations of order $O(h^3)$ with the pair $EQ_1^{rot}/Q_{10} \times Q_{01}$.

As a first attempt, in the present work, under weaker hyphethesis of $u_t \in L^2(0, T; H^2(\Omega))$ instead of $u_t \in L^2(0, T; H^3(\Omega))$ required in [30], we study a new mixed finite element scheme for the nonlinear Sobolev Eq. (1.1), and obtain the unconditional superclose and superconvergent results by avoiding the estimate of the numerical solution in L^∞-norm. It is worthy to be emphasized that, since our fully-discrete analysis is defined by an average of those at two consecutive time levels, the linearized system is much more technically complicated than that for the linearized backward Euler scheme in [20,21].

The rest of the paper is organized as follows. In Section 2, we introduce some notations and the mixed finite element scheme. In Section 3, the superclose results of order $O(h^2)$ for the original variable u in H^1-norm and the auxiliary variable $\vec{q} = -(a(u)\nabla u_t + b(u)\nabla u)$ in L^2-norm are deduced for the semi-discrete system. In Section 4, we give the linearized FEM for the fully-discrete system and derive the corresponding superclose estimates of order $O(h^2 + \tau^2)$, which are time-independent by estimating $\xi^n = \tau\sum_{i=2}^{n}\frac{\xi^n - \xi^{n-1}}{\tau} + \xi^1$ in a technical way. In Section 5, the new interpolated postprocessing operators are constructed which have smaller degrees of freedom than that of [28], and the corresponding global superconvergence results of order $O(h^2)$ and $O(h^2 + \tau^2)$ are deduced for semi-discrete and fully-discrete systems, respectively. In Section 6, some numerical results are provided to verify the theoretical analysis, and show the efficiency of the method.

2. Construction of mixed finite elements and preliminaries

Let T_h be a regular rectangular subdivision of Ω. $K \in T_h$ is an element with four vertices a_i and with four edges $l_i = \overline{a_i a_{i+1}}$, $i = 1, 2, 3, 4 \pmod 4$ parallel to x-axis and y-axis, $h = \max_K diam(K)$. The finite element spaces V_h and $\vec{W}_h$ are defined by

$$\begin{aligned} V_h &= \{v;\, v|_K \in Q_{11}(K), \forall K \in T_h\}, V_0^h = \{v;\, v \in V_h, v|_{\partial\Omega} = 0\},\\ \vec{W}_h &= \{\vec{w} = (w^1, w^2) \in (L^2(\Omega))^2;\, \vec{w}|_K \in Q_{01}(K) \times \in Q_{10}(K), \forall K \in T_h\}, \end{aligned}$$

where $Q_{ij} = span\{x^r y^s, 0 \le r \le i, 0 \le s \le j\}$.

For $v \in H^2(\Omega)$, $\vec{w}_h = (w_1, w_2) \in (H^1(\Omega))^2$, we define the associated interpolation operators I_h and Π_h as

$$I_h : v \in H^2(\Omega) \to I_h v \in V_h, I_h|_K = I_K, I_K v(a_i) = v(a_i), i = 1, 2, 3, 4,$$

and

$$\Pi_h : \vec{q} \in (H^1(\Omega))^2 \to \Pi_h\vec{q} \in \vec{W}_h, \Pi_h|_K = \Pi_K, \int_{l_i}(\vec{q} - \Pi_K\vec{q}) \cdot \vec{\tau}_i ds = 0,$$

respectively, where $\vec{\tau}_i$ is the unit tangent vector of l_i.

For $u \in H^3(\Omega)$, $\vec{q} \in (H^2(\Omega))^2$, there hold [28]

$$(\nabla(u - I_h u), \nabla v) \le Ch^2\|u\|_3\|v\|_1, \quad \forall v_h \in V_h, \tag{2.1}$$

$$(\vec{q} - \Pi_h\vec{q}, \vec{w}_h) \le Ch^2\|\vec{q}\|_2\|\vec{w}_h\|_0, \quad \forall \vec{w}_h \in \vec{W}_h. \tag{2.2}$$