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Quartic and quintic B-spline methods for advection–diffusion equation

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ABSTRACT

Differential quadrature methods based on B-spline functions of degree four and five have been introduced to solve advection–diffusion equation numerically. Two initial-boundary value problems modeling the transportation of a concentration and distribution of an initial pulse are simulated using both methods. The errors of the numerical results obtained by both methods have been computed. Stability analysis for both methods is also studied by the use of matrix stability.

1. Introduction

In industrialized world, a wide variety of contaminants are released to the environment everyday from industrial sources and plants. Modeling the contamination rate or transportation of contamination physically may probably the first step to solve an environment problem. The advection-diffusion equation (ADE) is a mathematical model for transport, dispersion, diffusion or intrusion in various media. Consider one-dimensional form of the ADE given by

$$\frac{\partial U}{\partial t} + \alpha \frac{\partial U}{\partial x} - \beta \frac{\partial^2 U}{\partial x^2} = 0, \quad 0 \le x \le L \tag{1}$$

with the initial condition

$$U(x, 0) = U_0(x), \quad 0 \le x \le L \tag{2}$$

and the boundary conditions

$$U(0, t) = f(t), \quad U(L, t) = g(t) \tag{3}$$

in a finite domain $[0, L]$ where α and β are parameters, $\frac{\partial U}{\partial x}$ and $\frac{\partial^2 U}{\partial x^2}$ are advection and diffusion terms, respectively [1]. In many environment problems, $U(x, t)$ represents concentration of the pollutant or contaminant material at point x at the time t. Sometimes, the solutions refer to mass, heat, water or energy transportation in various media containing draining film or soil [2–4]. In some studies, the ADE models many engineering and chemistry problems covering dispersion in porous media, the intrusion of fluids of different densities, the absorption of chemicals, dispersion of contaminants in rivers, lakes, embouchures and coasts, flow of a solute material through a tube, the transportation of pollutants in atmosphere, cooling problems in generators, the thermal pollution in water systems [5–13] etc. The ADE was solved numerically as a model in some financial forecasting problems [14].

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Differential Quadrature Method (DQM) is a derivative approximation technique constructed on the sum of the functional values at finite problem domain to solve partial differential equations numerically [15]. So far, many types of differential quadrature methods derived from various basis functions covering Lagrange interpolation polynomials, Hermite polynomials, radial basis functions, sinc functions and spline functions to solve many engineering and physics problems [16–37].

2. Numerical methods

Consider the uniform grid distribution $a = x_1 < x_2 < \cdots < x_N = b$ with N grid points of the interval $[a, b]$ and the functional values of $U(x, t)$ at each grid and time t are defined as $U(x_i, t)$, $i = 1, \ldots, N$. The approximation to derivatives of $U(x, t)$ with respect to the variable x is

$$\frac{\partial U^{(r)}(x_i,t)}{\partial x^{(r)}} = \sum_{j=1}^{N} w_{ij}^{(r)} U(x_j,t), \quad i = 1, 2, \ldots, N, \tag{4}$$

where $w_{ij}^{(r)}$ stands for the weighting coefficients of the approximation and r is the order of the derivative [15]. According to the method, before approximate to derivative terms, the weighting coefficients $w_{ij}^{(r)}$ should be computed by substituting the basis functions in Eq. (4) instead of $U(x_i, t)$. In this study, both quartic and quintic B-spline functions are considered to determine the weighting coefficients.

2.1. Quartic B-spline differential quadrature method (QRTDQ)

Let $\varphi_i(x)$ be a quartic B-spline. Then, the quartic B-spline functions set $\{\varphi_{-1}(x), \varphi_0(x), \ldots, \varphi_{N+1}(x)\}$ forms a basis set for functions defined over $[x_1 = a, b = x_N]$. The quartic B-spline functions $\varphi_i(x)$ are defined as:

$$\varphi_i(x) = \frac{1}{h^4}\begin{cases} (x-x_{i-2})^4, & [x_{i-2}, x_{i-1}] \\ (x-x_{i-2})^4 - 5(x-x_{i-1})^4, & [x_{i-1}, x_i] \\ (x-x_{i-2})^4 - 5(x-x_{i-1})^4 + 10(x-x_i)^4, & [x_i, x_{i+1}] \\ (x_{i+3}-x)^4 - 5(x_{i-2}-x)^4, & [x_{i+1}, x_{i+2}] \\ (x_{i+3}-x)^4, & [x_{i+2}, x_{i+3}] \end{cases} \tag{5}$$

where $h = x_i - x_{i-1}$ is the equal sub interval size [38]. All the functional values of the quartic B-spline $\varphi_i(x)$ outside the sub interval $[x_{i-2}, x_{i+3}]$ are zero. Substitution of each quartic B-spline function into the DQM equation (4) for a fixed x_i and r gives

$$\frac{d^{(r)}\varphi_m(x_i)}{dx^{(r)}} = \sum_{j=m-1}^{m+2} w_{ij}^{(r)} \varphi_m(x_j), \; m = -1, 0, \ldots, N+1, \tag{6}$$

The linear equation system (6) can be rewritten in the matrix notation as;

$$\begin{bmatrix} \varphi_{-1,-2} & \varphi_{-1,-1} & \varphi_{-1,0} & \varphi_{-1,1} & & \\ & \varphi_{0,-1} & \varphi_{0,0} & \varphi_{0,1} & \varphi_{0,2} & \\ & & \varphi_{N+1,N} & \varphi_{N+1,N+1} & \varphi_{N+1,N+2} & \varphi_{N+1,N+3} \end{bmatrix} \begin{bmatrix} w_{i,-2}^{(r)} \\ w_{i,-1}^{(r)} \\ w_{i,N+3}^{(r)} \end{bmatrix} = \Psi \tag{7}$$

for any x_i in the interval $[a, b]$. In the matrix form, $\varphi_{i,j}$ denotes $\varphi_i(x_j)$ and

$$\boldsymbol{\Psi} = \left[\frac{d^{(r)}\varphi_{-1}(x_i)}{dx^{(r)}}, \frac{d^{(r)}\varphi_0(x_i)}{dx^{(r)}}, \ldots, \frac{d^{(r)}\varphi_{N+1}(x_i)}{dx^{(r)}}\right]^T$$

Even though the linear equation system (7) contains $N + 3$ equations, it has $N + 6$ unknowns, namely $w_{ij}^{(r)}$, $j = -2, -1, \ldots, N + 3$. In order to reduce it to a solvable system, three more equations

$$\frac{d^{(r+1)}\varphi_{-1}(x_i)}{dx^{(r+1)}} = \sum_{j=-2}^{1} w_{i,j}^{(r)} \varphi'_{-1}(x_j)$$

$$\frac{d^{(r+1)}\varphi_{N}(x_i)}{dx^{(r+1)}} = \sum_{j=N-1}^{N+2} w_{i,j}^{(r)} \varphi'_{N}(x_j)$$

$$\frac{d^{(r+1)}\varphi_{N+1}(x_i)}{dx^{(r+1)}} = \sum_{j=N}^{N+3} w_{i,j}^{(r)} \varphi'_{N+1}(x_j)$$