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# A frequency-independent and parallel algorithm for computing the zeros of strictly proper rational transfer functions

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ABSTRACT

We develop an algorithm for computation of the zeros of a strictly proper rational transfer function in partial fraction form, by transforming the problem of finding the roots of the determinant of a frequency-dependent matrix into one of finding the eigenvalues of a companion matrix comprised of the determinants of a binomial-based set of frequency-independent matrices. The proposed algorithm offers a fundamentally new approach that avoids solving severely ill-conditioned system of linear equations, where condition numbers increase rapidly with frequency. The developed algorithm is straightforward, and enables parallel computation of the characteristic polynomial coefficients $a'_n$ that comprise the companion matrix to the characteristic polynomial $\Sigma_n a_n s^n$ of the frequency-dependent matrix. Additionally, the algorithm allows for relatively inexpensive computation of asymptotically accurate approximants of the transfer function, such that $a'_n$ need be computed only for selected powers of $s = \jmath\omega$, where the number of required determinant operations are shown to be relatively small. Additionally, limitations of the developed algorithm are highlighted, where the computational cost is shown to be on the order $\mathcal{O}(2^{N_p})$ determinant operations on matrices of dimensions $(N_p + 1) \times (N_p + 1)$, and $N_p$ is the number of poles. Illustrative numerical examples are selected and discussed to provide further insight about pros/cons of the proposed method, and to identify potential areas for further research.



## 1. Introduction

Sustaining the fast growing technology trends in wireless, mobile, low-power, ultra-thin, and high-speed microelectronic circuits and systems is critically contingent upon the accurate and efficient analysis, design, and optimization of the signal and power networks of the systems on chip and systems on package [1]; this requires accurate and stable frequency- and time-domain models and methodologies for signal and power integrity simulations [2–4].

Often, in signal and power integrity analysis, the transfer function (TF) of various elements, devices, and systems are expressed in terms of frequency-dependent multi-port network scattering parameters (S-parameters); this is the case for models of the interconnect of transmission lines [5] in an integrated circuit (IC) which connects the transistors on-die, to the package pins (or balls, pads, lands, etc.) on the printed circuit board (PCB).

Given the broadband character of high-speed digital signals propagating through the interconnect, the interconnect system model needs to cover a wide range of frequencies (typically from DC to tens of GHz). Due to the immense complexity in material

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and geometric properties of the interconnect system, its TF is usually obtained as numerical (tabulated) multi-port frequency-dependent S-parameters [6], either through measurement (e.g., Vector Network Analyzer), or through solving Maxwell's equations using field-solver software (e.g., [7]). Subsequently, the tabulated TF may be transformed into an equivalent circuit model (e.g., through vector fitting [8]) and attached to other device (SPICE) models that act as excitation or load to the interconnect system, and the entire system may be simulated to optimize the design.

In the process of signal and power integrity characterization, modeling, and simulation, the tabulated interconnect TF may be expressed as a rational function in the partial fraction form with poles and residues [9]. It is often desired (and sometimes necessary) to convert from this pole/residue form into the equivalent pole/zero form; examples of this include, system-zero identification, TF inversion [10] (e.g. impedance to admittance, etc.), equivalent circuit synthesis [11], stability analysis [12], root-causing of system interference problems, etc.

Previous investigations [8,13,14] have developed novel techniques for finding the poles/residues of the partial fraction form, primarily by minimizing the error between the partial fraction approximant and the tabulated data. In this work, we have three main contributions, as follows. (1) We develop an algorithm that operates on a *strictly* proper rational TF in pole/residue form, and produces the TF zeros; note that [8] only addresses the (non-strictly) proper rational TF. The proposed method is believed to be new, and provides an *exact* solution to the zeros, while (2) providing a convenient mechanism for polynomial truncation which enables accurate asymptotic approximation. (3) We present carefully selected numerical examples and provide discussion to highlight some of the important strengths and weaknesses of the proposed method, while pointing out potential areas for future research.

In Section 2, we provide some background information about linear time invariant (LTI) systems embodied as strictly proper rational transfer functions, in the partial fraction form. In Section 3 we develop the formulation that leads to the proposed algorithm. The numerical results are presented in Section 4, and discussion and conclusions are provided in Section 5.

## 2. Background

A linear time-invariant system [15] may be expressed in the usual matrix notation, as follows

$$\begin{aligned} \dot{x}(t) &= A \cdot x(t) + B \cdot u(t) \\ y(t) &= C \cdot x(t) + D \cdot u(t), \end{aligned} \tag{1}$$

where the system state is $x(t)$, its time derivative is $\dot{x} = \frac{d}{dt}x(t)$, the input to the system is $u(t)$, the output of the system is $y(t)$, and $A, B, C, D$ are time-invariant system matrices.

Taking the Laplace transform [16] of the above system, yields

$$\begin{aligned} sX(s) &= A \cdot X(s) + B \cdot U(s) \\ Y(s) &= C \cdot X(s) + D \cdot U(s), \end{aligned} \tag{2}$$

where $X(s)$, $Y(s)$, and, $U(s)$, are the Laplace transforms of $x(t)$, $y(t)$, and $u(t)$; respectively. The Laplace variable $s = \sigma + \jmath\omega$, where $\sigma$ is the attenuation term in units of (Hz), $\omega$ (rad/sec) is the angular frequency, and the imaginary number $\jmath = \sqrt{-1}$.

Solving for $X(s)$ in the first line of (2), and substitution in the second line of (2), results in

$$Y(s) = H(s) \cdot U(s), \tag{3}$$

where the system transfer function $H(s)$ is given by

$$H(s) = C \cdot (sI - A)^{-1}B + D \tag{4}$$

Note that the passive system TF (e.g. the transmission lines in microelectronic interconnect) in the Laplace domain, may be mathematically represented as a rational function [17,18] which is a ratio of two polynomials

$$H(s) = \frac{Q(s)}{P(s)} = \frac{a_0 + a_1 s + \cdots + a_{N_q-1}s^{N_q-1} + s^{N_q}}{b_0 + b_1 s + \cdots + b_{N_p-1}s^{N_p-1} + s^{N_p}}, \tag{5}$$

where the $\{N_q, N_p\} \in$ Integers, $\{a_n, b_n\} \in$ Reals, and the index $\{n, p, q\} \in$ Integers.

A strictly proper [19] rational function (i.e., $N_q < N_p$), may be written in the partial fraction form, as follows

$$H(s) = \frac{Q(s)}{P(s)} = \sum_{n=1}^{N_p} \frac{c_n}{s - p_n}, \tag{6}$$

where $c_n$ is the $n$th residue, and $p_n$ is the $n$th pole. It is assumed that all poles are distinct (i.e., $p_n \neq p_m$, $\forall m \neq n$ where $m \in$ Integers).

For real residues and poles, it is $\{c_n, p_n\} \in$ Real. For complex-conjugate pair of poles and residues, the TF is expressed by

$$H(s) = \frac{Q(s)}{P(s)} = \sum_{n=1}^{N_p} \frac{c_n}{s - p_n} + \frac{c_n^*}{s - p_n^*}, \tag{7}$$

where $c_n = c_{n_r} + \jmath c_{n_i}$, $p_n = p_{n_r} + \jmath p_{n_i}$, $x_{n_r}$ denotes the real part and $x_{n_i}$ denotes the imaginary part of $x_n$, $^*$ is the complex conjugate operator, $\{c_{n_r}, c_{n_i}, p_{n_r}, p_{n_i}\} \in$ Real, and all poles are assumed to be distinct (i.e., $p_{n_r} \pm \jmath p_{n_i} \neq p_{m_r} \pm \jmath p_{m_i}$, $\forall m \neq n$).