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Degenerate Mittag-Leffler polynomials

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ABSTRACT

We present several explicit formulas and recurrence relations for the degenerate Mittag-Leffler polynomials. We also give several connections between Mittag-Leffler polynomials and other known families of polynomials.

1. Introduction

The classical Mittag-Leffler polynomials $g_n(x)$ are given by ordinary generating function as

$$\sum_{n\geq 0} g_n(x)t^n = \left(\frac{1+t}{1-t}\right)^x. \tag{1}$$

with $|t| < 1$. The first few terms of them are $g_0(x) = 1$, $g_1(x) = 2x$, $g_2(x) = 2x^2$, $g_3(x) = \frac{4}{3}x^3 + \frac{2}{3}x$, $g_4(x) = \frac{2}{3}x^4 + \frac{4}{3}x^2$, $g_5(x) = \frac{4}{15}x^5 + \frac{4}{3}x^3 + \frac{2}{5}x$. They were introduced by Mittag-Leffler in an investigation of analytic representation of the integrals and invariants of a linear homogeneous differential equation (see [21]). One refers to the paper of Bateman for the summary of basic properties of them (see [2,3]). In fact, the Mittag-Leffler polynomials can be expressed in terms of the Gauss hypergeometric function ${}_2F_1$ as

$$g_n(x) = 2x\,{}_2F_1\left(\begin{matrix}1-n,\ 1-x\\ 2\end{matrix}\Big|2\right),$$

for all $n \geq 1$. In that sense, they can be considered as a special case of the Meixner polynomials $M_n(x;\beta,c)$ with $\beta = 2$ and $c = -1$ (see [17]). Namely, their relation is

$$g_n(x) = 2xM_{n-1}(x-1;2,-1).$$

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Also, the Mittag-Leffler polynomials satisfy the recurrence relation $(n+1)g_{n+1}(x) = (n-1)g_{n-1}(x) + 2xg_n(x)$ with $n \geq 1$, and orthogonality relation

$$\int_{-\infty}^{+\infty} g_n(-iy)g_m(iy)\frac{dy}{y\sinh(\pi y)} = \frac{2}{n}\delta_{n=m}, \quad n, m \geq 1.$$

There are several interesting researches on Mittag-Leffler polynomials and their properties and generalizations (for instance, see [2,3,21,25,26]). In particular, a recent study of these polynomials together with a generalization can be found in [26] and an application of the Mittag-Leffler polynomials to an expansion for the Riemann zeta function is discussed in [24]. Note that the Mittage-Leffler polynomials are connected with the *Pidduck polynomials* $P_n(x)$ by the expression $P_n(x) = \frac{1}{2}(e^{d/dx} + 1)g_n(x)$, where the Pidduck polynomials are defined by the generating function

$$\frac{t}{1-t}\left(\frac{1+t}{1-t}\right)^x = \sum_{n\geq 0} P_n(x)\frac{t^n}{n!}.$$

The aim of this paper to introduce the degeneration case of Mittag-Leffler polynomials, and study several properties of these polynomials. We also give several connections between Mittag-Leffler polynomials and other known families of polynomials.

Following Roman [22,23], we define Mittag-Leffler polynomials as $M_n(x) = n!g_n(x)$ so that

$$\sum_{n\geq 0} M_n(x)\frac{t^n}{n!} = \left(\frac{1+t}{1-t}\right)^x. \tag{2}$$

Then one can show that

$$M_n(x) = \sum_{k=0}^{n}\binom{n}{k}(n-1)_{n-k}2^k(x)_k,$$

where $(x)_n = x(x-1)(x-2)\ldots x-(n-1))$ with $(x)_0 = 1$ is the *falling factorial polynomials*. The degenerate Bernoulli polynomials, the first degenerate version of well-known families of polynomials, were introduced by Carlitz [5,6] and rediscovered by Ustinov [27] under the name of Korobov polynomials of the second kind. On the other hand, Korobov polynomials (of the first kind) are the degenerate version of the Bernoulli polynomials of the second kind (see [18,19]).

Recently, many researchers began to study various kinds of degenerate versions of the familiar polynomials like Bernoulli, Euler, falling factorial and Bell polynomials (see [9–14]) by using generating functions, umbral calculus, and p-adic integrals. Here we define the degenerate Mittag-Leffler polynomials and study their properties, and connections with other known families of polynomials by exploiting umbral calculus technique.

2. Definitions and preliminaries

As a degenerate version of the falling factorial polynomials $(x)_n$, the *degenerate falling factorial polynomials* $(x)_{n,\lambda}$ $(\lambda \neq 0)$ was introduced (see [11]) by the generating function

$$(1+\lambda)^{\frac{x}{\lambda}\frac{(1+t)^\lambda - 1}{\lambda}} = (1+\lambda)^{\frac{x((1+t)^\lambda-1)}{\lambda^2}} = \sum_{n\geq 0}(x)_{n,\lambda}\frac{t^n}{n!}. \tag{3}$$

Clearly, we have that $\lim_{\lambda\to 0}(x)_{n,\lambda} = (x)_n$, As to explicit expression for $(x)_{n,\lambda}$, the following are obtained

$$(x)_{n,\lambda} = \sum_{k=0}^{n}\left(\frac{\log(1+\lambda)}{\lambda}\right)^k S_1(n,k|\lambda)x^k,$$

where $S_1(n,k|\lambda)$ are the degenerate Stirling numbers of the first kind given by the generating function

$$\frac{1}{k!}\left(\frac{(1+t)^\lambda - 1}{\lambda}\right)^k = \sum_{n\geq k} S_1(n,k|\lambda)\frac{t^n}{n!}. \tag{4}$$

For convenience, we put $S_1(n,k|\lambda) = 0$ for all $0 \leq n < k$. Clearly, we have that $\lim_{\lambda\to 0} S_1(n,k|\lambda) = S_1(n,k)$, the ordinary Stirling number of the first kind, and one can show that

$$S_1(n,k|\lambda) = \sum_{m=k}^{n} S_1(n,m)S_2(m,k)\lambda^{m-k},$$

where $S_2(m,k)$ is the ordinary Stirling number of the second kind. Further, if we set

$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_1(n,k||\lambda)x^k,$$