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Approximate controllability of Riemann–Liouville fractional differential inclusions

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ABSTRACT

In this paper, by using fractional calculus, multi-valued analysis, semigroup theory and the fixed-point technique, we study the approximate controllability for a class of Riemann–Liouville fractional differential inclusions. An example is given to illustrate the application of the abstract results.

1. Introduction

In the past three decades, the field of fractional calculus and its applications have gained a lot of attention. Many fields such as physics, fluid mechanics, viscoelasticity, heat conduction in materials with memory, chemistry and engineering can be described by fractional differential equations. In recent years, notable contributions have been made in theory and applications of fractional differential equations, see the monographs of Kilbas et al. [7], Podlubny [18] and Zhou [31], the works [8,14] and the references therein. For the existence of mild solutions for fractional differential equations with the Caputo fractional derivative, we can refer to [3,25,32–34].

Controllability is a very important concept and plays an important role in both the deterministic and the stochastic control theories. There are several controlling concepts should be distinguished, for instance, exact controllability, optimal control and approximate controllability. In 2011, Wang and Zhou [28] studied the optimal control for a class of fractional differential systems, see also [15] and the references therein. Moreover, Exact controllability enables to steer the system to the arbitrary final state while approximate controllability means that the system can be steered to arbitrary small neighborhood of final state. To study the exact controllability when the nonlinear term is independent of the control function, Triggiani [27] assumed the controllability operator to have an induced inverse on a quotient space. But if the semigroup associated with the system is compact, then the controllability operator is also compact and its inverse does not exist if the state space H is infinite dimensional. Thus, the exact controllability is too strong in infinite dimensional spaces and the approximate controllability is more appropriate. As for the existence and controllability results for integer order differential equations and inclusions, we can refer to [1,2,9,11,12,19,23]. In the past few years, much attention has been paid to establish sufficient conditions for the existence of mild solutions for fractional differential equations and inclusions, the readers can refer to [16,29] for more details. The approximate controllability for fractional differential equations and inclusions with the Caputo derivative was wildly studied, see, for instance [4,13,20–22,24,26,30].

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Recently, by applying the Laplace transform and probability density functions, Zhou et al. [35] gave a suitable concept on mild solutions for the Riemann–Liouville fractional evolution equation

$$\begin{cases} {}^LD^q x(t) = Ax(t) + (Fx)(t), \quad t \in J' = (0, b], \\ (I_{0^+}^{1-q}x)(0) + g(x) = x_0, \end{cases} \tag{1.1}$$

where ${}^LD^q$ is the Riemann–Liouville fractional derivative of order q, $I_{0^+}^{1-q}$ is the Riemann–Liouville integral of order $1-q$, A is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Banach space X, $F: C(J', X) \to L(J', X)$ and $g: C(J', X) \to L(J', X)$ are given functions satisfying some assumptions. In [10], the authors studied the approximate controllability of impulsive fractional neutral evolution equations. However, the approximate controllability for Riemann–Liouville fractional differential inclusions is still rarely treated in the literature.

In this paper, we mainly discuss the approximate controllability for Riemann–Liouville fractional differential inclusions of the following form

$$\begin{cases} {}^LD^\alpha[x(t) - h(t, x(t))] \in Ax(t) + F(t, x(t)) + Bu(t), \quad t \in J' = (0, b], \\ I_{0^+}^{1-\alpha}[x(t) - h(t, x(t))]\,|_{t=0} = x_0 \in H, \end{cases} \tag{1.2}$$

where ${}^LD^\alpha$ is Riemann–Liouville fractional derivative of order $\frac{1}{2} < \alpha \leq 1$; $x(\cdot)$ takes values in a Banach space H; A: $D(A) \subset H \to H$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ in a Banach space H; the control function u takes its values in $L^2(J, U)$, U is a Banach space; B is a linear bounded operator from U to H; $F: J \times H \to \mathcal{P}(H) := 2^X H \setminus \emptyset$ is a nonempty, bounded, closed, and convex multivalued map; h: $J \times H \to H$ satisfies some assumptions given in (H_3).

The rest of this paper is organized as follows. In Section 2, we will give some notations and useful concepts about fractional calculus. In Section 3, we establish the existence of mild solutions for the system by using the fixed point theorem. In Section 4, the approximate controllability of the system is presented. In Section 5, we will give an example to illustrate the application of the abstract results. The conclusion is made in Section 6.

2. Preliminaries

In this section, we introduce some preliminary facts which are used throughout this paper. Let $(H, \|\cdot\|_H)$ denote a Banach space. For the uniformly bounded analytic semigroup $\{S(t)$: $t \geq 0\}$ in H, we assume that $\sup_{t\in[0,\infty)} \|S(t)\| := M < \infty$. Let $C(J, H)$ denote the Banach space of all H-valued continuous functions from $J = [0, b]$ to H with the norm $\|x\|_C = \sup_{t\in J} \|x(t)\|_H$. Let $J' = (0, b]$, to define the mild solution of system (1.2), we also consider the Banach space $C_{1-\alpha}(J, H) = \{x \in C(J', H) : t^{1-\alpha}x(t) \in C(J, H)\}$ with the norm

$$\|x\|_{C_{1-\alpha}} = \sup\{t^{1-\alpha}\|x(t)\|_H,\ t \in J'\}.$$

We need some basic definitions and properties about fractional calculus; essential principles of multi-valued analysis [5,6]; primary facts in semigroup theory [17] and some lemmas.

Definition 2.1 [32]**.** The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds,\ t > 0,\ \alpha > 0,$$

provided the right side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 [7]**.** The Riemann–Liouville fractional order derivative of order α with the low limit 0 for a function $f: [0, +\infty) \to R$ is defined as

$${}^LD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha+1-n}} ds,\ t > 0, n-1 < \alpha \leq n.$$

Definition 2.3 [29]**.** The Caputo's derivative of order α with the low limit 0 for a function $f: [0, +\infty) \to R$ can be written as

$${}^cD^\alpha f(t) = {}^L D^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right],\ t > 0, \quad n-1 < \alpha < n.$$

Lemma 2.1 [33]**.** *(Bochner's theorem) A measurable function* Q: $[0, a] \to E$ *is Bochner integrable if* $|Q|$ *is Lebesgue integrable.*

Throughout this paper, we assume that $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Then for $0 < \eta \leq 1$, it is possible to define the fractional power A^η as a closed linear operator on its domain $D(A^\eta)$ (see [33]). For analytic semigroup $\{S(t)\}_{t\geq 0}$, the following properties will be used:

(i) There is $M \geq 1$ such that

$$M := \sup_{t\in[0,+\infty)} S(t) < \infty; \tag{2.1}$$