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Theoretical analysis for blow-up behaviors of differential equations with piecewise constant arguments

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ABSTRACT

In this paper we discuss the blow-up behaviors of differential equations with piecewise constant arguments (EPCAs). Some fundamental results on the local existence and uniqueness of solutions of EPCAs are reviewed and some conditions are given under which the unique solution exists globally. Sufficient conditions for the finite blowup are presented and some examples illustrate that the blow-up behaviors of EPCAs are quite different from those of the corresponding ordinary differential equations.

1. Introduction

Initial value problems (IVPs) for ordinary differential equations (ODEs)

$$\begin{cases} x'(t) = f(t, x(t)), & t \geqslant t_0, \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

are an important mathematical tool for modeling an evolution system in many application problems. The finite blowups of ODEs have a physical meaning: the problem of thermal explosion, shock waves, thermal self-focusing beam structures in magneto-hydrodynamics, compression in gas dynamics, etc. The analysis began from 1898 by Osgood. Recently, the blow-up behavior of higher order ODEs is presented in [1,6–8], which also gives a new explanation to the collapse of bridges.

Many real-life problems actually influenced not only by their current situation, but also by some history information. Therefore a more refined approach is a differential equation with delay arguments

$$x'(t) = f(t, x(t), x(\tau(t))), \quad t \geqslant t_0, \tag{1.2}$$

where t_0 is an initial time, $f(t, x, y)$ is a continuously differentiable function and $\tau(t) \geqslant t_0 - \tau$ for some $\tau \geq 0$ is a delayed function. In [5], the blow-up behavior of solutions of delay differential equations with a constant delay is investigated, which is generalized by Volterra integral equations with delays in [16]. In [16], the influence of the delay lag and the initial values to the blow-up behaviors of delay differential equations with a kind of vanishing delay, i.e., a proportional delay is also studied. Another kind of vanishing delay is piecewise constant arguments, e.g., $\tau(t) = [t]$ and the corresponding equations are called EPCAs, where $[\cdot]$ is the greatest integer function. These kinds of equations are initially studied in [2,3], which include impulsive and loaded

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equations of control theory in [3,13]. There are also some papers illustrating some numerical methods for EPCAs such as [9–12] and all of solutions to EPCAs drawn in this paper are approximated by collocation methods.

In this paper, we consider EPCAs with the form

$$\begin{cases} x'(t) = f(x(t)) + g(x([t])), & t \geqslant 0, \\ x(0) = x_0, \end{cases} \tag{1.3}$$

where $x_0 > 0$ is an initial value and $f, g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable functions. We are interested with the influence of the delay term to the finite blowup of corresponding ODEs. Based on the discussion of the local existence and continuation of solutions of EPCAs in [15], we present some sufficient conditions to the global existence in Section 2. Especially, it is proved that solutions to pure nonlinear EPCAs never blow up in finite time, even they increase very faster and solutions to linear ODEs with a nonlinear digit feedback control never blow up in finite time. In Section 3, some sufficient conditions to blow-up solutions of EPCAs with a nonlinear function f are provided and then some examples illustrate that the blow-up behaviors of EPCAs are quite different from the corresponding ODEs.

2. Fundamental results

In this section, we review some fundamental results on the local existence and uniqueness of solutions to EPCAs in [14] and present some conditions under which the unique solution exists globally.

2.1. Existence and uniqueness

In this subsection, we consider EPCAs

$$\begin{cases} x'(t) = f(t, x(t), x([t])), & t \geqslant 0, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $d \geq 1$ is an integer, $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuously differentiable function and $[\cdot]$ denotes the greatest integer function.

Definition 2.1. A solution of (2.1) on $[0, T]$ for some $T > 0$ is a function $x(t)$ that satisfies the conditions:

(i) $x(t)$ is continuous on $[0, T]$.
(ii) The derivative $x'(t)$ exists at each point $t \in [0, T]$, with the possible exception of the point $t = n, n = 0, 1, 2, \ \ldots$, where one-sided derivatives exist.
(iii) (2.1) is satisfied on each interval $[n, n+1) \subset [0, T]$.

Since $t - [t]$ vanishes at each integer point, applying the results in [4] for delay differential equations with a vanishing delay in each interval $[n, n+1)$, one obtains the local existence, uniqueness and the properties of non-continuable solutions of EPCAs in each interval $[n, n+1)$. And then by the fundamental continuation of ODEs, the limit $x(n+1) = \lim_{t\to n+1-} x(t)$ exists whenever

$$\sup_{t\in[n,n+1)} \|x(t)\| < \infty.$$

Thus the solution is able to be continuable to the right hand of $t = n+1$. Repeating this process, one obtains a global solution or a non-continuable solution, which blows up in some interval $[[T], T)$.

Theorem 2.2 [15]**.** *Assume that Ω is an open subset of $\mathbb{R}^d$ and $f : \mathbb{R} \times \Omega \times \Omega \to \mathbb{R}^d$ is continuous and differentiable with respect to the second and third arguments. Then there exists a unique solution of (2.1) in $[0, T)$ for some $T > 0$.*

Theorem 2.3 [15]**.** *Assume that $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and differentiable with respect to the second and third arguments. Then the unique non-continuable solution on the maximal interval $[0, T)$ of existence satisfies either $T = \infty$ or*

$$\limsup_{t\to T-} \|x(t)\| = \infty.$$

Remark 2.4. It follows from Theorem 2.3 that for a smooth function f, solutions of (2.1) either exist globally or blow up in finite time. Thus our blow-up analysis is based on the following statements.

(i) A solution exists globally if and only if it is bounded in any finite interval.
(ii) A solution blows up in finite time if and only if the maximum interval of existence is finite.

Remark 2.5. Different from ODEs and delay differential equations with constant or proportional delays, the delay function $\tau(t) = [t]$ is a piecewise smooth function but not continuous at any integer point. Hence in general, the solution $x(t)$ is not differentiable at $t = n, n = 0, 1, 2, \ \ldots$. To overcome this, we always separate an interval $[0, T)$ into

$$[0, T) = \begin{cases} \cup_{n=0}^{[T]-2} [n, n+1) \cup [T-1, T), & T \text{ is an integer}, \\ \cup_{n=0}^{[T]-1} [n, n+1) \cup [[T], T), & \text{otherwise}. \end{cases}$$

In each subinterval the solution is smooth and all theorems in this paper are able to be proved by induction if necessary.