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Bayesian optimal control for a non-autonomous stochastic discrete time system

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ABSTRACT

The main objective of this article is to develop Bayesian optimal control for a class of non-autonomous linear stochastic discrete time systems. By taking into consideration that the disturbances in the system are given by a random vector with components belonging to an exponential family with a natural parameter, we prove that the Bayes control is the solution of a linear system of algebraic equations. For the case that this linear system is singular, we apply optimization techniques to gain the Bayesian optimal control. Furthermore, we extend these results to generalized linear stochastic systems of difference equations and provide the Bayesian optimal control for the case where the coefficients of this type of systems are non-square matrices.

1. Introduction

Linear stochastic discrete time systems (or linear matrix difference equations), are systems in which the variables take their value at instantaneous time points. Discrete time systems differ from continuous time ones in that their signals are in the form of sampled data. In real systems, the discrete time system often appears when it is the result of sampling the continuous-time system or when only discrete data are available for use. With the development of the digital computer, the stochastic discrete time system theory plays an important role in control theory.

When such systems are under consideration, performance measurement and the available information at the moments of control specification are two very important factors. The small deviations of the parameters can be treated as disturbances. As the random disturbance is admitted, the performance measure will be the mean value of the deviation of the states from the required behavior of the system. When all the parameters of the system are known and the distribution of disturbances is well defined, then the optimal control can be determined. The extension of this model to an adaptive one means that the disturbances are uncertain. Adaptive control is the control method used by a controller which must adapt to a controlled system with parameters which vary, or are initially uncertain (see Tesfatsion [36] for the history of the adaptive control).

Based on the behavior of the system we can learn the details of the disturbances. It is assumed that the disturbance has a fixed probabilistic description which is determined by the assumption. In this paper it is assumed that the distribution function is known to be an accuracy of parameters and the disturbances additionally change the state of the system. It resembles the statistical problem of estimation.

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Initially, it was the seminal papers by Wald [38,39], where the background of modern decision theory was established. The decision theory approach to control problems was then applied some years later (see books by Aoki [1], Sage and Melsa [29] and Sworder [34]). The new class of control systems under uncertainty was called *adaptive* (see Tesfatsion [36]). In these adaptive control problems the important role have Bayesian systems. In this class of control models it is assumed that the preliminary knowledge of the disturbances is given by *a priori* distributions of their parameters.

Bayes control is a very important chapter for these kinds of systems and thus many authors have studied it since then. Among others the topic was included to the monographs by Aoki [1], Ogata [22], Rao [26], Sandefur [30], and the papers by Kappen et al. [17], Kushner [18], Porosinski and Szajowski [23], Runggaldier [28], Sarkka and Sarmavuori [31], Tulsyan et al. [37].

The construction of the Bayes control is also auxiliary for the construction of minimax controls (see Szajowski and Trybuła [35], Porosiński and Szajowski [24], Grzybowski [16], González-Trejo et al. [15], Magiera [19]).

Stochastic discrete time systems have many applications in economics (see the papers by Dai et al. [2], Dassios and Zimbidis [3], Dassios et al. [4], Federico [13], Runggaldier [28] and Soner [33]), physics, circuit theory (see [12,14]), and other areas (see [1,22,26,30]). Recently, studies have been extended to fractional discrete time systems. The fractional nabla operator is a very interesting tool when applied to systems of difference equations, see [8–11] and has many applications especially in macroeconomics, since it succeeds to provide information from a specific year in the past until the current year.

Let us consider the following non-autonomous linear stochastic discrete time system (see [18])

$$\overline{x}_{n+1} = \alpha_n\overline{x}_n + b_n\overline{u}_n + c_n\overline{\upsilon}_n, \quad \forall n = 0, 1, \ldots, N-1. \tag{1}$$

Where $\overline{x}_n \in \mathbb{R}^m$ is the state of the system, $\overline{u}_n \in \mathbb{R}^m$ is the control, $\overline{\upsilon}_n \in V \subset \mathbb{R}^m$, with $\overline{\upsilon}_n = (\upsilon_n^1, \upsilon_n^2, \ldots, \upsilon_n^k, 0, \ldots, 0)^T$, is the disturbance at time n and $\alpha_n, b_n, c_n \in \mathbb{R}^{m\times m}$. We assume that the disturbances υ_n^i belong to an exponential family with parameter λ_i, $i = 1, 2, \ldots, k$, i.e. the disturbance $\overline{\upsilon}_n$ belongs to an exponential family with parameter $\overline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k, 0, \ldots, 0)^T$. The horizon N of the control, the time up to which the system is controlled, is assumed to be fixed and independent of the disturbances $\overline{\upsilon}_n$, $n \geq 0$. Note that even if the horizon was assumed to be a random variable, i.e. to have the known distribution

$$P\{N = k\} = p_k, \quad \forall k = 0, 1, \ldots, M, \quad \sum_{i=0}^{M} p_k = 1, \quad p_M \neq 0,$$

a problem with a random horizon can always be converted to a problem with fixed horizon. Let

$$X_n = (\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_n) \tag{2}$$

and

$$U_n = (\overline{u}_0, \overline{u}_1, \ldots, \overline{u}_n), \quad U^n = (\overline{u}_n, \overline{u}_{n+1}, \ldots, \overline{u}_N). \tag{3}$$

Definition 1.1. For convenience U_N will be denoted by U and called control policy.

Definition 1.2. We define the cost of control for a given control policy U (the loss function) as

$$L(U, X_N) = \sum_{i=0}^{N} (\overline{y}_i^T s_i\overline{y}_i + \overline{u}_i^T k_i\overline{u}_i). \tag{4}$$

Where $k_i \in \mathbb{R}^{m\times m} \geq 0_{m,m}$, are symmetric matrices, $s_i \in \mathbb{R}^{2m\times 2m} \geq 0_{2m,2m}$ and $\overline{y}_i = \left(\overline{x}_i \cdots \overline{\lambda}\right) \in \mathbb{R}^{2m}$, $\forall i = 0, 1, \ldots, N$. With $0_{i,j}$ we will denote the zero matrix $i \times j$.

Definition 1.3. (See [18,25,32]) Let $E_N, E_{\overline{\lambda}}$ be the expectations with respect to the distributions of N and $\overline{\upsilon}_n$ when $\overline{\lambda}$ is the parameter. For an a priori distribution π of $\overline{\lambda}$, let E_π, E be the expectations with respect to the distribution π and to the joint distribution $\overline{\upsilon}_n$ and $\overline{\lambda}$, respectively. Then if $L(U, X_N)$ is the loss function defined by (4):

(a) The risk connected to the control policy U, when the parameter $\overline{\lambda}$ is given, is defined as follows

$$R(\overline{\lambda}, U) = E_N\left[E_{\overline{\lambda}}[L(U, X_N) \mid X_0]\right] = E_N\left[E_{\overline{\lambda}}[\overline{y}_i^T s_i\overline{y}_i + \overline{u}_i^T k_i\overline{u}_i \mid X_0]\right];$$

(b) The expected risk r, associated with π and the control policy U, is equal to

$$r(\pi, U) = E_\pi[R(\overline{\lambda}, U)] = E_N\left[E[\overline{y}_i^T s_i\overline{y}_i + \overline{u}_i^T k_i\overline{u}_i \mid X_0]\right];$$

(c) The expected risk is given by

$$r(\pi, U^n) = E_N\left[E[\overline{y}_i^T s_i\overline{y}_i + \overline{u}_i^T k_i\overline{u}_i \mid X_n, U_{n-1}] \mid N \geq n\right].$$

Where X_n is defined by (2) and U_n, U^n are defined by (3).

The paper is organized as follows: Section 2 provides some remarks on disturbances. In Section 3 we determine the Bayes control for the conjugate prior distribution π of the parameter $\overline{\lambda}$ as the solution of a singular linear system and provide the optimal Bayesian control. We close the paper by studying the Bayes control of a class of generalized linear stochastic discrete time systems.