Applied Mathematics and Computation 274 (2016) 745–761

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

An explicit analytic approximation of solutions for a class of neutral stochastic differential equations with time-dependent delay based on Taylor expansion

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ARTICLE INFO

MSC:
60H10

Keywords:
Neutral stochastic differential equations
Time-dependent delay
Taylor approximation
L^p and almost sure convergence

ABSTRACT

This paper represents a contribution to the analysis of approximate methods for stochastic differential equations based on the application of Taylor expansion, under the Lipschitz and linear growth conditions. The L^p and almost sure convergence of the appropriate approximate solutions are considered for a class of neutral stochastic differential equations with time-dependent delay. Coefficients of the approximate equations, including the neutral term, are Taylor approximations of the coefficients of the initial equation up to the first derivatives. For $p \geq 2$, the rate of the L^p-convergence of the sequence of approximate solutions to the exact solution is estimated as $\frac{2lp-1}{2l}$, where $l > 1$ is an integer. The presence of the neutral term in the equation reflected to the rate of convergence.

1. Introduction and preliminary results

The application of Taylor expansion is a well-known approach to the approximation of solutions of different types of stochastic differential equations. Numerous approximate methods are based on the Taylor approximations of coefficients of stochastic differential equations, which can be found, for example, in [1–3] and in the literature cited therein. Moreover, in the literature, one can find convergence results related to analytical approximate methods based on the application of Taylor expansion (see, for example, [6–10]). As one can observe from the previously cited papers, the order of the L^p-closeness between the exact solution and appropriate approximate solution increases when the number of terms in the Taylor approximations of coefficients of equation increases.

The main motivation for this paper is to implement the same approach to a class of neutral stochastic differential equations with time-dependent delay, bearing in mind that the analysis of such equations often requires the application of non-trivial techniques, as one can observe from [11–15]. Precisely, the main goal was to obtain large enough rate of the L^p-convergence knowing, for example from [14,15], that the rate of the L^2-convergence of the Euler–Maruyama method for neutral stochastic functional and neutral stochastic differential equations with time-dependent delay is $\frac{l-1}{l}$, where $l > 1$ is an integer.

The initial assumption is that all random variables and processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq 0}, P)$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is increasing and right-continuous, and $\mathcal{F}_0$ contains all P-null sets). Let $w(t) = (w_1(t), w_2(t), \ldots, w_m(t))^T, t \geq 0$ be an m-dimensional standard Brownian motion, $\mathcal{F}_t$-adapted and

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http://dx.doi.org/10.1016/j.amc.2015.11.026

independent of $\mathcal{F}_0$. Let the Euclidean norm be denoted by $|\cdot|$ and matrix norm by $\text{trace}[B^T B] = \|B\|^2$, where B^T is the transpose of a vector or a matrix.

For a given $\tau > 0$, let $C([-\tau, 0]; R^d)$ be the family of continuous functions φ from $[-\tau, 0]$ to R^d, equipped with the supremum norm $||\varphi||_c = \sup_{-\tau \le \theta \le 0} |\varphi(\theta)|$.

Let us begin with discussion of the following equation in which the delay is time-dependent. In that sense, we present a Borel measurable function δ: $[t_0, T] \to [0, \tau]$ such that

$$d[x(t) - u(x(t - \delta(t)))] = f(x(t), x(t - \delta(t)))dt + g(x(t), x(t - \delta(t)))dw(t), \quad t \in [t_0, T], \tag{1}$$

$$x_{t_0} = \xi = \{\xi(\theta), \theta \in [-\tau, 0]\}, \tag{2}$$

where u: $R^d \to R^d$,f: $R^d \times R^d \to R^d$ and g: $R^d \times R^d \times \to R^{d \times m}$ are Borel measurable and $x(t)$ is a d-dimensional state process. The initial condition ξ is supposed to be a $\mathcal{F}_{t_0}$-measurable and $C([-\tau, 0]; R^d)$-valued random variable for which $E\|\xi\|_c^p < \infty,\ p \ge 2$.

It should be emphasized that the approach which will be used in this paper, was initially used in the context of ordinary stochastic differential equations in the papers [4,5] by Atalla and later, in papers [6–10], it is generalized and extended to different classes of more complex stochastic differential equations. In those papers it has been shown, for different classes of stochastic differential equations, that the rate of the L^p-convergence of the appropriate approximate solutions to the solution of the initial equation increases when the number of terms in Taylor series increases.

To the best of the author's knowledge, there is no available literature which deals with the Taylor approach in the framework of neutral stochastic differential equations with time-dependent delay. So, in order to extend results from the previously cited papers, we want to approximate the solution of Eq. (1) using a sequence of solutions to neutral stochastic differential equations with neutral term, drift and diffusion coefficients which are Taylor approximations of u, f and g, up to the first derivatives. In that sense, the presence of the neutral term in Eq. (1) requires a special treatment which presents the main difference between the present paper and the previously cited papers. Among other things, the consequence of the presence of the neutral term is that we will consider the Taylor expansion of coefficients of Eq. (1) up to the first derivative, in order to obtain certain order of the L^p-convergence.

2. Main results

Let us first present Eq. (1) in its equivalent integral form, that is, for $t \in [t_0, T]$,

$$x(t) = \xi(0) + u(x(t - \delta(t))) - u(x(t_0 - \delta(t_0))) + \int_{t_0}^{t} f(x(s), x(s - \delta(s)))ds + \int_{t_0}^{t} g(x(s), x(s - \delta(s)))dw(s), \tag{3}$$

with the initial condition (2). Let also $t_0 < t_1 < \cdots < t_n = T$ be a partition of the interval $[t_0, T]$ where n is chosen in a way that there exists an integer n_* such that $\tau = n_* \frac{T - t_0}{n}$. Moreover, we suppose that n is large enough integer such that $\delta_n = (T - t_0)/n \in (0, 1)$. So, the partition points of the interval $[t_0 - \tau - \delta_n, T]$ are

$$t_k = t_0 + \frac{k}{n}(T - t_0), \quad k = -n_* - 1, -n_*, -n_* + 1, \ldots, -1, 0, 1, \ldots, n, \tag{4}$$

where an additional partition point t_{-n_*-1} is added for the purpose of the definition of the following approximate method. In that sense, we set $\delta(t_{-1}) = \delta(t_0)$.

Our goal is to approximate the solution of Eq. (3) on the partition (4) by the process $\{x^n(t) : t \in [t_0 - \tau - \delta_n, T]\}$ using Taylor expansion of the coefficients u, f and g. In order for this solution to be well defined, we set

$$x^n(t) = x^n(t + \delta_n) = \xi(t + \delta_n - t_0), \quad t \in [t_0 - \tau - \delta_n, t_0 - \tau] \tag{5}$$

and require that the approximate solution x^n satisfies the initial condition

$$x^n(t) = \xi(t - t_0), \quad t \in [t_0 - \tau, t_0]. \tag{6}$$

If we denote by $\lfloor \cdot \rfloor$ the integer part function, then, for any $t \in [t_k, t_{k+1}]$, $k \in \{0, 1, \ldots, n-1\}$, define the linear interpolation $z_k^n(t)$ of $x^n(t_{k-1-\lfloor \delta(t_{k-1})/\delta_n \rfloor})$ and $x^n(t_{k-\lfloor \delta(t_k)/\delta_n \rfloor})$ as

$$z_{k,j}^n(t) = x_j^n(t_{k-1-\lfloor \delta(t_{k-1})/\delta_n \rfloor}) + \frac{t - t_k}{\delta_n}\left(x_j^n(t_{k-\lfloor \delta(t_k)/\delta_n \rfloor}) - x_j^n(t_{k-1-\lfloor \delta(t_{k-1})/\delta_n \rfloor})\right), \ j \in \{1, 2, \ldots, d\} \tag{7}$$

and $z_k^n(t) = (z_{k,1}^n(t), z_{k,2}^n(t), \ldots, z_{k,d}^n(t))$.

Then, the approximate solution $\{x^n(t), t \in [t_k, t_{k+1}]\}$, $k = 0, 1, \ldots, n-1$ is determined by the equations

$$\begin{aligned} x^n(t) &= x^n(t_k) + U_1(x_{t-\delta(t)}^n; z_k^n(t)) - U_2(x_{t_k - \delta(t_k)}^n; x_{t_{k-1-\lfloor \delta(t_{k-1})/\delta_n \rfloor}}^n) \\ &\quad + \int_{t_k}^{t} F(x_s^n, x_{s-\delta(s)}^n; x_{t_k}^n, x_{t_{k-\lfloor \delta(t_k)/\delta_n \rfloor}}^n)\, ds + \int_{t_k}^{t} G(x_s^n, x_{s-\delta(s)}^n; x_{t_k}^n, x_{t_{k-\lfloor \delta(t_k)/\delta_n \rfloor}}^n)\, dw(s), \ \ t \in [t_k, t_{k+1}], \end{aligned} \tag{8}$$

satisfying initial conditions $x_{t_k}^n = x^n(t_k + \theta)$, $\theta \in [-\tau, 0]$ a.s. for $k = 1, 2, \ldots, n-1$, that is (6), together with (5), for $k = 0$, where

$$U_1(x_{t-\delta(t)}^n; z_k^n(t)) = u(z_k^n(t)) + D^{(1)}u(z_k^n(t))\Delta z_k^n(t), \tag{9}$$