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Milne method for solving uncertain differential equations

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ABSTRACT

Uncertain differential equation is a type of differential equation driven by Liu process and has been widely applied to many fields especially to uncertain finance. Unfortunately, the analytic solutions of uncertain differential equations cannot always be obtained. So far, some numerical methods have been investigated. This paper designs a new numerical algorithm for solving uncertain differential equations via Milne method.

1. Introduction

Probability theory has been developing for a long time with being used to model random phenomena. A fundamental assumption of using probability theory is that the available distribution and the real frequency are close enough. That is, if we want to use probability we should possess large historical data. But in real research programs, it is hard for us to obtain enough data via statistics to estimate the probability distribution. In this case, we have to invite some experts to evaluate the belief degrees about the chances of the possible events happening. Liu [15] showed that human beings usually estimate a much wider range of values than the object actually takes. That is, the conservative estimation of belief degrees has a much larger range than the real frequency. Hence, the belief degree could not be modeled by probability measure.

Naturally, Liu [5] founded uncertainty theory using uncertain measure to deal with the belief degree in 2007 and Liu [7] perfected it with presenting product uncertain measure. Meanwhile, uncertain variable, a basic concept, was proposed by Liu [5]. For describing the uncertain variable, Liu [5] introduced a concept of uncertainty distribution. Then Peng and Iwamura [17] gave a sufficient and necessary condition for the uncertainty distribution of an uncertain variable. Additionally, Liu [7] raised a concept of independence with respect to uncertain variables, based on which the operational law was established by Liu [9]. For ranking uncertain variables, a concept of expected value was put forward in [5]. In addition, the independence of uncertain vectors was discussed by Liu [13]. Based on the uncertainty theory, some important and useful results were obtained such as Li and Liu [4], Liu [8,10,11].

In 2008, Liu [6] first gave a concept of uncertain process for modeling the evolution of uncertain phenomena. For describing the uncertain process, a concept of uncertainty distribution was introduced by Liu [14]. Meanwhile, Liu [14] provided a sufficient and necessary condition for the uncertainty distribution of an uncertain process. Besides, Liu proposed the independence of uncertain processes, based on which the operational law for uncertain processes was put forward by Liu [14] and a concept of stationary independent increment process was introduced by Liu [6]. Then Liu [7] proposed a canonical Liu process, which is an uncertain process with stationary and independent normal uncertain increments. Meanwhile, uncertain calculus was established by Liu [7] to deal with the integral and differential of an uncertain process in regard to canonical Liu process.

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Liu first proposed uncertain differential equation in [6] which is a type of differential equations driven by Liu process defined by Chen and Ralescu [2]. Then Chen and Liu [1] provided a sufficient condition for an uncertain differential equation having a unique solution. After a while, Gao [3] gave an existence and unique theorem under weaker conditions. Then, Liu [7] presented a concept of stability for an uncertain differential equation in the sense of uncertain measure. Then Yao et al. [22] gave a sufficient condition for stability. After that, Yao et al. [23] proposed a concept of stability in mean, and gave a sufficient condition. In addition, Sheng and Wang [19] studied another type of stability in the sense of pth moment.

Besides obtaining some properties, we have derived analytic solutions of some special types of uncertain differential equations. Chen and Liu [1] solved a linear uncertain differential equation, and obtained its analytic solution. Following that, Liu [16] provided some analytic methods to solve two special types of nonlinear uncertain differential equations. In addition, Yao and Chen [20] proposed Yao–Chen formula which may efficiently determine the inverse uncertainty distribution of the solution of an uncertain differential equation. Furthermore, uncertain differential equations have been applied into other fields successfully. For example, Liu [7] first applied the uncertain differential equation to dealing with a stock model in uncertain market. Additionally, Zhu [24] first introduced the uncertain differential equation into optimal control problems.

Unfortunately, it is usually difficult to find analytic solutions of uncertain differential equations. Hence, we must design some numerical methods to solve uncertain differential equations. Yao [21] designed a numerical algorithm for solving an uncertain differential equations via Euler method. Shen and Yao [18] used Runge–Kutta method to resolve uncertain differential equations. This paper aims at providing a new numerical method to solve uncertain differential equations via Milne method.

The remainder of this paper is organized as follows. Some basic concepts and properties will be reviewed in Section 2, including uncertain variable, uncertain process, uncertain integral and uncertain differential equation. We will devote Section 3 to presenting Milne method and some numerical experiments. We will apply Milne method to extreme values and integrals of solutions of uncertain differential equations, and present some corresponding numerical examples in Sections 4 and 5 respectively. Finally, we will discuss conclusions in Section 6.

2. Preliminaries

In this section, we will introduce some fundamental concepts and properties concerning uncertain variables, uncertain processes, and uncertain differential equations.

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element Λ in $\mathcal{L}$ is called an event and assigned a number $\mathcal{M}\{\Lambda\}$ to indicate the belief degree with which we believe Λ will happen. In order to deal with belief degrees rationally, Liu [5] suggested the following three axioms:

Axiom 1 (Normality Axiom)**.** $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;

Axiom 2 (Duality Axiom)**.** $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ;

Axiom 3 (Subadditivity Axiom)**.** For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots,$ we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

Definition 1 (Liu [5])**.** The set function $\mathcal{M}$ is called an uncertain measure if it satisfies the normality, duality, and subadditivity axioms.

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Furthermore, the product uncertain measure on the product σ-algebra $\mathcal{L}$ was defined by Liu [7] as follows:

Axiom 4 (Product Axiom)**.** Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots,$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrary events chosen from $\mathcal{L}_k$ for $k = 1, 2, \ldots,$ respectively.

Definition 2 (Liu [5])**.** An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \,|\, \xi(\gamma) \in B\}$$

is an event.

Theorem 1. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be uncertain variables, and f a real-valued measurable function. Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable defined by*

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \ldots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma.$$