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Stability and exponential stability of linear discrete systems with constant coefficients and single delay

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ABSTRACT

This paper investigates the exponential stability and exponential estimate of the norms of solutions to a linear system of difference equations with single delay

$$x(k+1) = Ax(k) + Bx(k-m), \quad k = 0, 1, \dots$$

where A, B are square constant matrices and $m \in \mathbb{N}$. Sufficient conditions for exponential stability are derived using the method of Lyapunov functions and its efficiency is demonstrated by examples.

1. Introduction

Recently, the investigation of the properties of linear difference systems with delay has been receiving much attention [2–10,13–15,17–20].

In this paper, we deal with the exponential stability of linear difference systems with constant coefficients and single delay

$$x(k+1) = Ax(k) + Bx(k-m), \quad k = 0, 1, \dots \tag{1}$$

where A, B are $n \times n$ constant matrices, $x = (x_1, \dots, x_n)^T : \{-m, -m+1, \dots\} \to \mathbb{R}^n$, and $m \in \mathbb{N}$. Exponential-type stability and an exponential estimate of the norm of solutions are derived. Recall that the initial problem for the system (1) is determined by $(m+1)$ given constant vectors $x(-m)$, $x(-m+1)$, …, $x(0)$. We use the norm $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ for a vector x. Denote by $\rho(A)$ the spectral radius of the matrix A, by $\lambda_{\max}(\mathcal{A})$, $\lambda_{\min}(\mathcal{A})$ the maximum and minimum eigenvalues of a symmetric matrix $\mathcal{A}$ and set $\varphi(\mathcal{A}) := \lambda_{\max}(\mathcal{A})/\lambda_{\min}(\mathcal{A})$. For a matrix $\mathcal{B}$, we use the norm $|\mathcal{B}| = (\lambda_{\max}(\mathcal{B}^T\mathcal{B}))^{1/2}$. Throughout the paper, we assume $|A| + |B| > 0$, i.e., we assume that (1) is not degenerated, and we do not explicitly mention this property in the formulations of results.

A trivial solution $x(k) = 0$, $k = -m, -m+1, \dots$ of (1) is Lyapunov stable if, for arbitrary $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, for any other solution $x(k)$, we have $|x(k)| < \varepsilon$ for $k = 0, 1, \dots$ and $\|x(0)\|_m < \delta(\varepsilon)$ where $\|x(0)\|_m := \max\{|x(i)|, i = -m, -m+1, \dots, 0\}$. If, in addition, $\lim_{k\to+\infty} |x(k)| = 0$, the trivial solution is called asymptotic stable. The trivial solution of system (1) is called Lyapunov exponentially stable if there exist constants $N > 0$ and $\theta \in (0, 1)$ such that, for an arbitrary solution

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$x = x(k)$ of (1),

$$|x(k)| \le N\|x(0)\|_m \theta^k, \quad k = 1, 2, \ldots .$$

The notions of asymptotic stability and exponential stability are equivalent in the case of systems with constant coefficients (1) and the stability (exponential stability) of a trivial solution is equivalent to the stability (exponential stability) of any other solution. For the foundations of stability theory to difference equations, we refer, e.g., to [1,11,16].

Asymptotic stability of (1) can be investigated through the properties of the roots of the related characteristic equation. Since this equation is, in general, a polynomial equation of degree $(m+1)n$, a major difficulty in the investigation of the properties of its roots is that the order of the polynomial is high for large m. This circumstance is difficult to overcame since the application of the Schur-Cohn criterion [11,12] to verify that all roots lie inside the unit circle is not easy because of the computational difficulties.

We investigate the exponential stability of (1) by the second Lyapunov method and use suitable Lyapunov functions. We use the well-known assertion [11] that the linear system

$$x(k+1) = Ax(k), \quad k = 0, 1, \ldots$$

is exponentially stable if and only if, for an arbitrary positive definite symmetric $n \times n$ matrix C, the matrix equation

$$A^T H A - H = -C \tag{2}$$

has a unique solution - a positive definite symmetric matrix H.

For systems of differential equations with delay

$$\dot{x}(t) = f(x(t), x(t-\tau))$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\tau > 0$, the use of the finite Lyapunov functions involves finding a positive definite function $V(x)$ such that its total derivative along the solutions of the system

$$\frac{d}{dt} V(x(t)) = \mathrm{grad}^T V(x(t)) \cdot f(x(t), x(t-\tau))$$

is negative definite provided that the solutions approach from the inside the surface level $V(x) = \alpha$ for a positive constant α. This is called a Razumikhin condition and can be written as

$$V(x(s)) < \alpha = V(x(t))$$

where $t - \tau \le s < t$. For a quadratic Lyapunov function $V(x) = x^T H x$, we obviously have

$$\lambda_{\min}(H)|x|^2 \le V(x) \le \lambda_{\max}(H)|x|^2, \tag{3}$$

and the Razumikhin condition implies

$$\lambda_{\min}(H)|x(t-\tau)|^2 \le V(x(t-\tau)) < V(x(t)) \le \lambda_{\max}(H)|x(t)|^2$$

or

$$|x(t-\tau)| < \sqrt{\varphi(H)}|x(t)|.$$

Below, we consider a similar approach to studying the stability of difference systems with delay (1). The rest of the paper is organized as follows. In Section 2, we derive sufficient conditions for Lyapunov stability. In Section 3, the exponential stability of system (1) and exponential estimates of solutions are investigated. Concluding remarks and examples demonstrating results obtained are considered in Section 4.

2. Lyapunov stability

In this part, to prove Lyapunov stability, we utilize the quadratic Lyapunov function $V(x) = x^T H x$. Define auxiliary numbers

$$L_1(H) := \lambda_{\max}(H) - \lambda_{\min}(C) + |A^T H B|,$$
$$L_2(H) := \lambda_{\min}(H) - \varphi(H)\left[|A^T H B| + |B^T H B|\right],$$
$$L_3(H) := \lambda_{\min}(C) - \left(1 + \varphi^2(H)\right)|A^T H B| - \varphi^2(H)|B^T H B|.$$

Sufficient conditions for the stability of (1) are given by the following theorem.

Theorem 1. *Let $\rho(A) < 1$, C be a fixed positive definite symmetric $n \times n$ matrix, matrix H solves the corresponding Lyapunov matrix equation (2), and*

$$L_i(H) > 0, \quad i = 1, 2, 3. \tag{4}$$

Then, the system with delay (1) is Lyapunov stable.