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Feedback control for some solutions of the sine-Gordon equation

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ABSTRACT

Evolution of an initial localized bell-shaped state for the sine-Gordon equation is considered. It is obtained numerically that variation in the parameters of the localized input gives rise to different propagating waves as time goes. The speed gradient feedback control method is employed to achieve unified wave profile weakly dependent on initial conditions. Two speed-gradient like algorithms are developed and compared. It is shown that the algorithm using coefficient at the second spatial derivative term in the sine-Gordon equation allows one to generate the same wave with prescribed energy from different initial states having different energies.

Introduction

It is known that solutions to nonlinear equations are sensitive to initial conditions, and even moderate variations in them result in a qualitatively different wave evolution. One of them is the sine-Gordon equation that describes many interesting phenomena, e.g., dynamics of coupled pendulums, Josephson junction arrays, interaction of atomic chains [1–3], it also accounts for continuum limits of the crystalline lattices [4–6].

One possibility to recover the unified shape of the wave may be in applications of the control methods [7–9]. Methods of control theory (cybernetics) attract a growing interest of physicists for more than two decades [9–11]. Among many effects achievable by means of control is reducing sensitivity on initial conditions. Previously these methods were mainly used for oscillation control problems for systems governed by ordinary differential equations [9,12]. Use of the control mechanism in the wave processes concern envelope wave equations [7,13], reaction-diffusion equations [14,15], and the sine-Gordon equation [8,16,17]. Some control related methods for sin-Gordon equation use feedforward (nonfeedback) controlling actions [7,13]. The other ones apply control changing the equation completely, and do not using measurement of the current system state [14,15]. A method for asymptotic stabilization of the sine-Gordon equation without damping by high-gain output boundary feedback is proposed in [16,17]. However efficient methods to control oscillatory modes in the sine-Gordon equation were not proposed, according to the best authors' knowledge. Therefore it is interesting to study a possibility to control oscillatory modes of the sine-Gordon equation by means of the speed-gradient method that has been successfully applied to ODE oscillatory systems [9].

The control may be established by different ways. For example, the control of dispersive terms in the equations may be performed [7]. However, the feedback control methods look more promising. A successive attempt to extend the control methods

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to the wave problems has been done in [8,16] where the boundary conditions were controlled to achieve significant difference in the wave behavior of the sine-Gordon equation and some of its generalization.

In this paper, we develop an algorithm of control with a feedback for evolution of the bell-shaped input for the sine-Gordon equation and compare it with the previous algorithm suggested in [9]. The difference between the algorithms is that they control coefficients at different terms of the sine-Gordon equation. Control of one or another coefficient may have a physical reason of it may be in variation of a spring rigidity if application to the dynamics of coupled pendulums is considered [1,3].

The paper is organized as follows. First section is devoted to development of the speed gradient algorithms with a feedback based on a control of different coefficients of the sine Gordon equation. Next section considers evolution of an initial localized bell-shaped input without control. Application of the control algorithms is studied in Section 3, while conclusions summarize the results and discuss future work.

1. Speed-gradient algorithms for the control with a feedback of the sine-Gordon equation

Consider a solution $U(x, t)$ of the sine-Gordon equation,

$$U_{tt} - U_{xx} + \sin(U) = 0, \tag{1}$$

One of the speed-gradient control algorithms with a feedback has been developed for the sine-Gordon equation in [9]. It was suggested there to include an external action,$F = F(t)$, in the equation yielding

$$U_{tt} + F\sin U - U_{xx} = 0, \quad t \geq 0, \tag{2}$$

Then it was assumed that $F = 1 + u(t)$, where $u(t)$ is a control action. The aim of the control is supposed to achieve the basic energy of system (4) equal to the defined value H^*,

$$H(t) \to H^*. \tag{3}$$

The Hamiltonian for Eq. (1) is

$$H = \frac{1}{2}\int_{-\infty}^{+\infty} \left(U_t^2 + U_x^2 + (1 - \cos U)\right) dx. \tag{4}$$

Then the control of the speed-gradient $u = -\gamma \dfrac{\partial \omega}{\partial u}$ was obtained in [9]

$$u(t) = \gamma\left(H(t) - H^*\right)\int_{-\infty}^{+\infty} U_t \cdot \sin U \, dx, \tag{5}$$

where $\gamma > 0$ is a parameter of amplification of the algorithm.

Another algorithm may be developed similarly for the control of the coefficient at U_{xx},

$$U_{tt} + \sin U - (1 + u(t))U_{xx} = 0, \quad t \geq 0, \tag{6}$$

where $u = u(x, t)$ is a control action. Following the procedure from [9] one obtains that temporal variation of the energy (2) is not zero due to the control action,

$$\begin{aligned} \frac{dH}{dt} &= \int_{-\infty}^{+\infty} (U_t \cdot U_{tt} - U_{xx} \cdot U_t + \sin U \cdot U_t)\, dx = \int_{-\infty}^{+\infty} U_t \cdot (U_{tt} - U_{xx} + \sin U)\, dx \\ &= \int_{-\infty}^{+\infty} U_t \cdot \left(- U_{xx} + (1 + u)U_{xx}\right) dx = u(t)\int_{-\infty}^{+\infty} U_t \, U_{xx}\, dx. \end{aligned} \tag{7}$$

Let us introduce the objective function $V(t) = \frac{1}{2}(H(t) - H^*)^2$, whose minimum corresponds to the satisfaction of the given condition $H(t) = H^*$. The control algorithm is developed using the speed-gradient method [9] relative to the function $V(t)$. Assuming $H^* = const_t$, one obtains $\omega(t) \equiv \frac{dV}{dt} = (H(t) - H^*)\frac{dH}{dt}$.

Taking partial derivative in u we obtain the speed-gradient control in the form

$$u(t) = -\gamma\left(H(t) - H^*\right)\int_{-\infty}^{+\infty} U_t \cdot U_{xx}\, dx. \tag{8}$$

One can note that control (8) may be more physically reasonable than control (5). In particular, the control of the coefficient at U_{xx} in Eq. (1) concerns variation in a spring rigidity if the sine-Gordon equation is used to describe the dynamics of coupled pendulums [1,3].

2. Evolution of localized input without feedback

Consider the case when initial condition for U has the form of a localized bell-shaped initial state (for example, in the form of the Gaussian distribution),