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A new method of dynamical stability, i.e. fractional generalized Hamiltonian method, and its applications

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ABSTRACT

In the paper, we present fractional generalized Hamiltonian method of dynamical stability, in terms of Riesz–Riemann–Liouville derivative, and study its applications. For an actual dynamical system, the fractional generalized Hamiltonian method of constructing a fractional dynamical model is given, and then the six criterions for fractional generalized Hamiltonian method of dynamical stability are presented. As applications, by using the fractional generalized Hamiltonian method, we construct five kinds of actual fractional dynamical models, which include a fractional Euler–Poinsot model of rigid body that rotates with respect to a fixed-point, a fractional Hojman–Urrutia model, a fractional Lorentz–Dirac model, a fractional Whittaker model and a fractional Robbins–Lorenz model, and we explore the dynamical stability of these models, respectively. This work provides a general method for studying the dynamical stability of an actual fractional dynamical system that is related to science and engineering.

1. Introduction

With the development of science and technology, the Hamiltonian dynamics has been playing an important role not only in mathematics, but also in engineering science, nonlinear science, mechanics, physics and so on [1–9]. But traditional Hamiltonian system is defined in even dimensional space where the structure has good characteristics, but it is limited in applications. In 1953, Pauli [10] studied the problems of quantized non-local field theory, and found additional support for the Hamiltonian mechanics. In 1959, Martin [11] was trying to promote the methods of a Hamiltonian to apply it to a system with a non-existing Lagrangian and got a backing for the Hamiltonian mechanics. Their results as an extension theory of the Hamiltonian mechanics were called the dynamics of generalized Hamiltonian systems. Since then, the study of the generalized Hamiltonian mechanics has been developed in theories and applications [12–14]. The problem of dynamical stability is a very essential and crucial issue for science and engineering, and has attracted many researchers [15–24]. However, the fractional generalized Hamiltonian method of dynamical stability is not presented so far.

Fractional dynamical method not only can more truly reveal the natural phenomena, but also is more close to the engineering practice. In the end of the 1970s, Mandelbrot discovered a fact that a large number of fractional dimensional examples exist in nature and engineering, and fractional calculus became a useful and important tool for researching fractal geometry [25]. This important discovery caused the shock of science, and scientists began to study many problems about the dynamical system with fractional derivatives. Since then, the study of the basic theories and methods for fractional dynamics has become a hot topic, and

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won wide development in theories and applications [26–39]. Recently, we established the fractional generalized Hamiltonian mechanics, which include its gradient representation, Lie algebraic structure, generalized Poisson conservation law, variation equations, construction method of integral invariants, and so on [40–43]. In order to better solve the fractional dynamical stability problems in science and engineering, it is necessary to propose the fractional generalized Hamiltonian method of dynamical stability.

In the paper, we present a new method, i.e. fractional generalized Hamiltonian method, for dynamical stability, and study its applications. Section 2 explains briefly the fractional generalized Hamiltonian system, and provides a fractional generalized Hamiltonian method of constructing fractional dynamical model. In Section 3, we present the fractional generalized Hamiltonian method of dynamical stability, and give its six criterions. Section 4 discusses special cases. In Applications A–E of Sections 5–9, by using the fractional generalized Hamiltonian method, we construct five kinds of fractional dynamical models, and we explore the dynamical stability of these models, respectively. Section 10 contains the conclusions.

2. Fractional generalized Hamiltonian method of constructing fractional dynamical model

Let us consider a fractional generalized Hamiltonian system of which the local coordinate of generalized Poisson manifold $(M, \{\cdot,\cdot\})$ is (x_1, x_2, x_3). The generalized Hamiltonian is $H = H(t, x_i)$, the structure element of a generalized Poisson bracket $\{\cdot,\cdot\}$ is $J_{ij}(x)=\{x_i, x_j\}$, fractional derivative ${}_a^R D_b^\alpha x$ is n times continuously differentiable in interval $[a, b]$, n is a positive integer and $\alpha > 0$. If the equations of motion of a fractional dynamical system can be expressed in the following form [40]:

$$ {}_a^R D_b^\alpha x_k = \frac{d}{dt}\left({}_a^R D_b^{\alpha-1} x_k\right) = \left\{{}_a^R D_b^{\alpha-1} x_k, H\right\} = J_{ij}\frac{\partial {}_a^R D_b^{\alpha-1} x_k}{\partial x_i}\frac{\partial H}{\partial x_j}, \quad (i, j, k = 1, 2, \ldots, m; n-1 \le \alpha < n), \tag{1} $$

then the system is called the fractional generalized Hamiltonian system with Riesz–Riemann–Liouville derivative. If generalized Hamiltonian H does not explicitly depend on time t, Eq. (1) is called the autonomous fractional generalized Hamiltonian system. Here $J_{ij}(x)$ meets the following conditions:

$$ J_{ij} = -J_{ji}, \tag{2} $$

$$ \sum_{l=1}^{m}\left[J_{il}\frac{\partial J_{jk}}{\partial x_l} + J_{jl}\frac{\partial J_{ki}}{\partial x_l} + J_{kl}\frac{\partial J_{ij}}{\partial x_l}\right] = 0, \quad (i, j, k = 1, 2, \ldots, m). \tag{3} $$

For an actual dynamical system, if we can construct its generalized Hamiltonian H and structure matrix $J_{ij}(x)$, then by using fractional generalized Hamiltonian Eq. (1), the fractional dynamical model of this system can be established. This method is called fractional generalized Hamiltonian method of constructing a fractional dynamical model.

3. Fractional generalized Hamiltonian method of dynamical stability

Suppose fractional generalized Hamiltonian system (1) has solutions

$$ {}_a^R D_b^{\alpha-1} x_k = {}_a^R D_b^{\alpha-1} x_k^0(t), \quad (k = 1, 2, \ldots, m), \tag{4} $$

and substituting Eq. (4) into Eq. (1), we have

$$ {}_a^R D_b^\alpha x_k^0(t) = \left(J_{ij}\right)_0\left(\frac{\partial {}_a^R D_b^{\alpha-1} x_k}{\partial x_i}\right)_0\left(\frac{\partial H}{\partial x_j}\right)_0, \quad (i, j, k = 1, 2, \ldots, m), \tag{5} $$

where $()_0$ means that $({}_a^R D_b^{\alpha-1} x_k)^0$ takes the place of ${}_a^R D_b^{\alpha-1} x_k$. Take Eq. (4) as the undisturbed motion, and let

$$ {}_a^R D_b^{\alpha-1} x_k = {}_a^R D_b^{\alpha-1} x_k^0(t) + {}_a^R D_b^{\alpha-1}\xi_k, \quad (k = 1, 2, \ldots, m), \tag{6} $$

then substituting Eq. (6) into Eq. (1), we can obtain the disturbed equations

$$ {}_a^R D_b^\alpha x_k^0(t) + {}_a^R D_b^\alpha \xi_k = \left(J_{ij}\right)_1\left(\frac{\partial {}_a^R D_b^{\alpha-1} x_k}{\partial x_i}\right)_1\left(\frac{\partial H}{\partial x_j}\right)_1, \quad (i, j, k = 1, 2, \ldots, m), \tag{7} $$

where $()_1$ means that ${}_a^R D_b^{\alpha-1} x_k^0 + {}_a^R D_b^{\alpha-1}\xi_k$ takes the place of ${}_a^R D_b^{\alpha-1} x_k$. Expanding $(J_{ij})_1$, $(\frac{\partial {}_a^R D_b^{\alpha-1} x_k}{\partial x_i})_1$ and $(\frac{\partial H}{\partial x_j})_1$ into Taylor series near equilibrium positions, we have

$$ \left(J_{ij}\right)_1 = \left(J_{ij}\right)_0 + \sum_{\rho=1}^{m}\left(\frac{\partial J_{ij}}{\partial {}_a^R D_b^{\alpha-1} x_\rho}\right)_0 {}_a^R D_b^{\alpha-1}\xi_\rho + \cdots, \tag{8} $$

$$ \left(\frac{\partial {}_a^R D_b^{\alpha-1} x_k}{\partial x_i}\right)_1 = \left(\frac{\partial {}_a^R D_b^{\alpha-1} x_k}{\partial x_i}\right)_0 + \sum_{\rho=1}^{m}\left(\frac{\partial^2\left({}_a^R D_b^{\alpha-1} x_k\right)}{\partial x_i \partial {}_a^R D_b^{\alpha-1} x_\rho}\right)_0 {}_a^R D_b^{\alpha-1}\xi_\rho + \cdots, \tag{9} $$