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Efficient Chebyshev collocation methods for solving optimal control problems governed by Volterra integral equations

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ABSTRACT

The main purpose of this work is to provide efficient Chebyshev collocation methods for solving optimal control problems (OCPs) governed by Volterra integral equations. The basic principle of our approach is to approximate the state and control using the Chebyshev polynomials and collocate the dynamic constraints at the Chebyshev-type points. Furthermore, we present an exact, efficient, and stable approach for computing the associated Chebyshev integration matrices. Numerical results on benchmark OCPs demonstrate the spectral rate of convergence for the proposed methods.

1. Introduction

Optimal control problems (OCPs) arise naturally in various areas of science, engineering, and mathematics. Considerable work has been done in the area of classic OCPs whose dynamics are described by ordinary differential equations. Recently, a class of OCPs whose dynamics are described by Volterra integral equations (VIEs) has received more and more attention. It is well known that VIEs can be used to model a variety of phenomena such as population dynamics, spread of epidemics, and continuum mechanics of materials with memory, to name a few but three.

By far, different numerical methods have been proposed for solving OCPs governed by VIEs, which, in general, can be grouped into two major categories: indirect methods and direct methods. In an indirect method, necessary optimality conditions of an OCP governed by VIEs are derived by using the calculus of variations, leading to a multiple-point boundary value problem that is then solved to obtain candidate optimal solutions. The related results can be found in, e.g., [1–7]. In a direct method, a continuous OCP governed by VIEs is transcribed to a finite-dimensional nonlinear programming problem (NLP) through the parameterization of the state and/or control variables in some manner, and the resulting NLP is then solved using well-known optimization software. The related results can be found in, e.g., [8–11]. It is also noteworthy to point out recent interest in developing numerical methods for solving integral and integro-differential equations [12–27].

The motivation of this paper is to provide a new direct method for the numerical solution of OCPs governed by VIEs using a spectral collocation approach. More precisely, we develop efficient Chebyshev collocation methods using collocation at the Chebyshev-type points to convert an OCP governed by VIEs into a NLP. Furthermore, we present an exact, efficient, and stable approach for computing the associated Chebyshev integration matrices (CIMs), and extend the proposed methods to solving OCPs governed by Volterra integro-differential equations (VIDEs).

The rest of this paper is organized as follows. In Section 2 the Chebyshev polynomials are presented for subsequent developments. The OCP governed by VIEs/VIDEs is described in Section 3. In Section 4 the proposed method with collocation at the

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Chebyshev–Gauss (CG) points is derived in detail. The computation of first-order CIM is provided in Section 5. Numerical results on three benchmark OCPs are shown in Section 6. Finally, Section 7 contains some concluding remarks.

2. Chebyshev polynomials

The Chebyshev polynomials of the first kind are orthogonal polynomials on the interval $[-1, +1]$, and satisfy the following orthogonality relation:

$$\int_{-1}^{+1} \omega_T(\tau) T_i(\tau) T_j(\tau)\,\mathrm{d}\tau = \int_{-1}^{+1} \frac{T_i(\tau)T_j(\tau)}{\sqrt{1-\tau^2}}\,\mathrm{d}\tau = \lambda_{T_i}\delta_{ij} \tag{1}$$

where $\omega_T(\tau) = \frac{1}{\sqrt{1-\tau^2}}$ is the Chebyshev weight function, δ_{ij} is the Kronecker delta function, and λ_{T_i} is the normalization factor, given by

$$\lambda_{T_i} = \begin{cases} \pi, & i = 0 \\ \dfrac{\pi}{2}, & i \neq 0 \end{cases} \tag{2}$$

The three-term recursion formula for the Chebyshev polynomials is given by

$$T_0(\tau) = 1, \quad T_1(\tau) = \tau, \tag{3a}$$

$$T_{n+1}(\tau) = 2\tau T_n(\tau) - T_{n-1}(\tau), \quad n = 1, 2, \ldots \tag{3b}$$

Some important properties of the Chebyshev polynomials are given by Canuto et al. [28]

$$T_n(\pm 1) = (\pm 1)^n \tag{4a}$$

$$|T_n(\tau)| \leq 1 \tag{4b}$$

$$T_n(-\tau) = (-1)^n T_n(\tau) \tag{4c}$$

$$T_n(\tau) = \begin{cases} 1 & n = 0 \\ n2^n \displaystyle\sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m (n-m-1)!}{2^{2m+1} m!(n-2m)!}\,\tau^{n-2m}, & n = 1, 2, \ldots \end{cases} \tag{4d}$$

$$T_n(\tau) = \begin{cases} 1, & n = 0 \\ \dfrac{1}{2(n+1)} T'_{n+1}(\tau) = \dfrac{1}{4} T'_2(\tau), & n = 1 \\ \dfrac{1}{2(n+1)} T'_{n+1}(\tau) - \dfrac{1}{2(n-1)} T'_{n-1}(\tau), & n = 2, 3, \ldots \end{cases} \tag{4e}$$

where $\lfloor n/2 \rfloor$ denotes the largest integer less than or equal to $n/2$.

3. OCP governed by VIE/VIDE

3.1. OCP governed by VIE

Consider the following general OCP governed by VIE. Determine the state, $\boldsymbol{x}(t) \in \mathbb{R}^{n_x}$, control, $\boldsymbol{u}(t) \in \mathbb{R}^{n_u}$, initial time, $t_0 \in \mathbb{R}$, and final time, $t_f \in \mathbb{R}$, on the time interval $t \in [t_0, t_f]$ that minimize the cost functional

$$J = \phi(\boldsymbol{x}(t_0), t_0, \boldsymbol{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\boldsymbol{x}(t), \boldsymbol{u}(t), t)\,\mathrm{d}t \quad \in \mathbb{R}, \tag{5}$$

subject to the dynamic constraints

$$\boldsymbol{x}(t) = \boldsymbol{h}(t) + \int_{t_0}^{t} \boldsymbol{f}(t, \boldsymbol{x}(s), \boldsymbol{u}(s), s)\,\mathrm{d}s \quad \in \mathbb{R}^{n_x}, \tag{6}$$

the inequality path constraints

$$\boldsymbol{c}(\boldsymbol{x}(t), \boldsymbol{u}(t), t) \leq \boldsymbol{0} \quad \in \mathbb{R}^{n_c}, \tag{7}$$

and the boundary conditions

$$\boldsymbol{b}(\boldsymbol{x}(t_0), t_0, \boldsymbol{x}(t_f), t_f) = \boldsymbol{0} \quad \in \mathbb{R}^{n_b}. \tag{8}$$