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A generalized Newton method for absolute value equations associated with circular cones

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ABSTRACT

In this paper, we study a class of absolute value equations associated with circular cone (CCAVE for short), which is a generalization of the absolute value equations discussed recently in the literature, analogously to the fact that circular cone is the very generalization of the second-order cone. We show that the CCAVE is equivalent to a class of circular cone linear complementarity problems, and hence generalize the well-known equivalence between absolute value equations and linear complementarity problems. Useful properties of the generalized differential of the absolute value function over the circular cone are investigated, which helps to propose a generalized Newton method for solving the CCAVE. The convergence of this method is established under mild conditions, as well as the efficiency of which is illustrated by some preliminary numerical results.

1. Introduction

The target problem of this paper is the following absolute value equations associated with circular cones (CCAVE for short):

$$Ax + B|x| = b, \tag{1}$$

where $A, B \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are given data; $x = (x_1^T, \cdots, x_r^T)^T \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$ is a variable with $n_1 + \cdots + n_r = n$; and $|x|$ denotes the absolute value of x which is determined by the corresponding circular cones and will be given in detail in the following section. Let $m, n, r, n_1, \cdots, n_r$ be positive integers. The Cartesian product $\mathcal{L}_\theta$ of circular cones $\mathcal{L}_\theta^{n_i}$'s (CC for short) is denoted as

$$\mathcal{L}_\theta := \mathcal{L}_\theta^{n_1} \times \cdots \times \mathcal{L}_\theta^{n_r},$$

where every $\mathcal{L}_\theta^{n_i}$ denotes a *circular cone* in $\mathbb{R}^{n_i}$ defined by

$$\begin{aligned}\mathcal{L}_\theta^{n_i} &:= \{x_i = (x_{i1}, x_{i2}) \in \mathbb{R} \times \mathbb{R}^{n_i-1} \mid \|x_i\| \cos\theta \le x_{i1}\} \\ &:= \{x_i = (x_{i1}, x_{i2}) \in \mathbb{R} \times \mathbb{R}^{n_i-1} \mid \|x_{i2}\| \le x_{i1} \tan\theta\}\end{aligned} \tag{2}$$

with $\|\cdot\|$ denoting the usual Euclidean norm of a vector. The circular cone [23] is a pointed closed convex cone having hyperspherical sections orthogonal to its axis of revolution about which the cone is invariant with respect to rotation. The circular cone is

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the very generalization of the second-order cone which is a powerful tool to handle optimization problems, and the relations between circular cone and second-order cone are as follows:

$$\mathcal{L}_\theta = A^{-1}\mathcal{K}^n \quad \text{and} \quad \mathcal{K}^n = A\mathcal{L}_\theta \quad \text{with} \quad A = \begin{bmatrix} \tan\theta & 0 \\ 0 & I \end{bmatrix},$$

which is proved in [23], Theorem 2.1. Likewise, for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, there have

$$x \in \mathcal{L}_\theta \Longleftrightarrow Ax \in \mathcal{K}^n, \qquad y \in \mathcal{L}_\theta^* \Longleftrightarrow A^{-1}y \in \mathcal{K}^n. \tag{3}$$

The standard absolute value equations (AVEs for short) in $\mathbb{R}^n$ study the system

$$Ax + B|x| = b, \quad \text{where } |x| := (|x_1|, |x_2|, \cdots, |x_n|)^T \in \mathbb{R}^n,$$

which were first introduced by Rohn [22] and are capable to formulate many optimization problems [12,16,20]. Recently, the AVEs have been extensively studied, see [6,7,10,13,15–19] and references therein. For the above AVEs and the CCAVEs (1), with the definition of the absolute value function over the circular cone (cf. (8)), it is easy to see that the CCAVEs (1) reduces to the SOCAVEs when $\theta = \frac{\pi}{4}$ which be studied in [7], and the CCAVEs (1) reduces to the AVEs when $n_1 = n_2 = \cdots = n_r = 1$ and $\theta = \frac{\pi}{4}$. Henceforth, the CCAVEs (1) is a generalization of the previous AVEs and the SOCAVEs. More interestingly, it will be shown (cf. Theorem 2.1) that the CCAVEs (1) are equivalent to the following circular cone linear complementarity problems (CCLCP for short): find the elements $z, w \in \mathbb{R}^n$ such that

$$\begin{aligned} & Mz + Pw = b \\ & z \in \mathcal{L}_\theta, \quad w \in \mathcal{L}_\theta^*, \quad \langle z, w\rangle = 0, \end{aligned} \tag{4}$$

where $M, P \in \mathbb{R}^{m\times n}$ and $c \in \mathbb{R}^m$ are given matrices and vector, respectively. We know that the complementarity problem plays a fundamental role in optimization theory and has many applications in engineering and economics [3,4,7–9,11]. Therefore, it would be worth exploring the absolute value equations, and it is one of the motivations of this work.

Towards to solutions of the AVEs, various numerical methods for solving the AVEs were proposed in the literature (see [2,7,13,14,25] and references therein). In this paper, we are interested in a generalized Newton method for solving the CCAVEs with $m = n$. More specifically, we show that the generalized Newton method is well-defined under an assumption that the all singular values of the matrix A exceed the maximal singular value of the matrix B. Furthermore, under suitable conditions, we show that this proposed method is globally linearly and locally quadratically convergent. In addition, we present some preliminary numerical results of the proposed method for solving the CCAVEs (1) to illustrate the efficiency of the proposed method.

The remaining parts of this paper are organized as follows. In Section 2, we give the explicit formula of projection of $x \in \mathbb{R}^n$ onto $\mathcal{L}_\theta$ and $\mathcal{L}_\theta^*$. Some concepts and preliminary results on the circular cone are given in this section. We study the Generalized Jocobian of the absolute value function $|x|$ and its properties in Section 3. In Section 4, we propose a generalized Newton method for solving the CCAVE (1), and discuss the convergence of the proposed method under suitable conditions. In Section 5, the preliminary numerical results are given.

2. Preliminaries

In this section, we briefly review some basic concepts and background materials about circular cones, which will be extensively used in the subsequent analysis.

As defined in (3), the circular cone $\mathcal{L}_\theta$ has a *revolution axis* which is the ray generated by the canonical vector $e_1 := (1, 0, \ldots, 0)^T \in \mathbb{R}^n$, and its dual cone $\mathcal{L}_\theta^*$ is easily seen as

$$\mathcal{L}_\theta^* := \{y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|y_2\| \sin\theta \le y_1\}.$$

Note that the circular cone $\mathcal{L}_\theta$ is not a self-dual cone when $\theta \neq \frac{\pi}{4}$, that is, $\mathcal{L}_\theta^* \neq \mathcal{L}_\theta$ whenever $\theta \neq 45°$. Therefore, $\mathcal{L}_\theta$ is not a symmetric cone for $\theta \in \left(0, \frac{\pi}{2}\right)\backslash\{\frac{\pi}{4}\}$. It is also known from [23,24] that the dual cone of $\mathcal{L}_\theta$ can be expressed as

$$\mathcal{L}_\theta^* = \{y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|y_2\| \le y_1 \cot\theta\} = \mathcal{L}_{\frac{\pi}{2}-\theta}.$$

In the next section, we talk about the projections onto $\mathcal{L}_\theta$ and $\mathcal{L}_\theta^*$. To this end, we let x_+ denote the projection of x onto the circular cone $\mathcal{L}_\theta$, and x_- be the projection of $-x$ onto the dual cone $\mathcal{L}_\theta^*$. With these notations, for any $x \in \mathbb{R}^n$, it can be verified that $x = x_+ - x_-$. Moreover, due to the special structure of the circular cone $\mathcal{L}_\theta$, the explicit formula of the projection of $x \in \mathbb{R}^n$ onto $\mathcal{L}_\theta$ is obtained in [23] as follows:

$$x_+ = \begin{cases} x & \text{if } x \in \mathcal{L}_\theta, \\ 0 & \text{if } x \in -\mathcal{L}_\theta^*, \\ u & \text{otherwise}, \end{cases} \tag{5}$$