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Asymptotic periodicity for hyperbolic evolution equations and applications

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ABSTRACT

We study the existence and uniqueness of pseudo S-asymptotically ω-periodic mild solutions for hyperbolic evolution equations. We show concrete applications of our methods.

1. Introduction

In the literature, we have studied several concepts to represent the idea of approximately periodic function. The asymptotic periodicity problem is one of the most active subjects in the behavior theory of solution of differential equations at present. We observe that concrete systems usually are submitted to external perturbations which are not periodic. In many real situations we can assume that these perturbations are approximately periodic in a broad sense. It has recently emerged the notion of S-asymptotically ω-periodicity (see [29]) which is a generalization of the classical asymptotically ω-periodicity. This new notion has interesting applications in several branches of differential equations. This has motivated considerable interest in the topic. We refer the reader to [14,17–19,24,39]. In addition, in [37] a new space of S-asymptotically ω-periodic functions was introduced. It is called the space of pseudo S-asymptotically ω-periodic functions ($PSAP_\omega$ in short). In that work the authors have studied qualitative properties of this type of functions. Also they discuss the existence of pseudo S-asymptotically ω-periodic mild solutions for abstract neutral functional equations. Some applications involving ordinary and partial differential equations with delay are presented. In [21] the authors have studied the existence of this type of solutions for a class of abstract fractional differential equations. The theory of integrations and derivatives of arbitrary order appear frequently in various research areas of sciences and engineering (see [23]). In Section 6.1 we apply our methods to fractional evolution equations.

We study in this work some sufficient conditions for the existence and uniqueness of a pseudo S-asymptotically ω-periodic mild solution to the following semilinear evolution equation:

$$u'(t) = Au(t) + f(t, u(t)),\ t \in \mathbb{R}, \tag{1.1}$$

with A: $D(A) \subseteq X \to X$ is the generator of a hyperbolic semigroup on X and f is a suitable continuous function. The development of this theory is directly connected with the theory of semigroups. The existence of pseudo S-asymptotically ω-periodic (mild) solutions for (1.1) remains an untreated topic in the literature. A key aspect of this study is developing an efficient composition result for this type of functions and hence with the help of hyperbolic semigroup theory and suitable fixed point theorems to obtain $PSAP_\omega$ solutions.

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Evolution equations of hyperbolic type are an active field of current research. Hyperbolic equations attract the attention of both mathematicians and physicists in view of its applications to real models. We observe that many authors have considered linear hyperbolic evolution equations in the Hilbert spaces framework (see [4–10]).

The contributions of this paper are two: (1) to develop an asymptotic periodicity theory to hyperbolic evolution equations and (2) to develop a connection between our set of abstract methods and a broad class of concrete problems.

We now turn to a summary of this work. Section 2 provides the definitions and preliminary results to be used in theorems stated and proved in the article. In particular to facilitate access to the individual topics, we review in Sections 2.1 and 2.2 some of the standard properties of S-asymptotically ω-periodic functions, which are the *Leitmotiv* of this work. Section 2.3 deals with hyperbolic semigroups and intermediate spaces. In Section 3 we obtain very general results on the existence of pseudo S-asymptotically ω-periodic (PSAP_ω in short) mild solutions for the semilinear problem (1.1) under Lipschitz type hypothesis on the nonlinearity. Also we consider complementary assumptions on the nonlinear term. In Section 4 we treat the special case where the operator A in (1.1) generates a uniformly stable semigroup, while that in Section 5 we study PSAP_ω-mild solutions of (1.1) in intermediate spaces. Finally, in Section 6, we apply our theory to concrete applications including fractional equations. We present several remarks and examples, leading to a better understanding of the work and hence attract the attention to researchers who are entering the subject.

2. Technical tool

In this section we collect some definitions, notations and properties that we use throughout this work. Let X and Y be two Banach spaces and let $I \subseteq \mathbb{R}$ be a (possibly unbounded) interval, $C_b(I; X)$ denotes the space formed by the bounded continuous functions from I into X endowed with the norm of uniform convergence. The notation $C_0(\mathbb{R}^+; X)$ stands for the subspace of $C_b(\mathbb{R}^+; X)$ consisting of functions that vanish at infinity. Moreover, if $X = \mathbb{R}$ or $\mathbb{C}$, we shall write $C_b(I)$, $C_0(\mathbb{R}^+)$ instead of $C_b(I, \mathbb{R}),\ C_0(\mathbb{R}^+; \mathbb{R})$. $C^\beta(I; X)$, $\beta \in (0, 1)$, denotes the space of all bounded continuous X-valued functions u on I such that $\|u\|_\beta = \sup\{\frac{\|u(s)-u(t)\|}{|s-t|^\beta} : s, t \in I, s \neq t\} < +\infty$. We put $\|u\|_{C^\beta} := \|u\|_\infty + \|u\|_\beta$. Then $C^\beta(I; X)$ is a Banach space under the norm $\|\cdot\|_{C^\beta}$. This set is called the space of the Hölder continuous functions. In this work $\mathcal{B}(X; Y)$ denotes the space of bounded linear operators from X into Y endowed with the uniform operator norm denoted by $\|\cdot\|_{\mathcal{B}(X;Y)}$, and we abbreviate the notation to $\mathcal{B}(X)$ and $\|\cdot\|_{\mathcal{B}(X)}$ whenever $X = Y$. For a closed linear operator B, we denote by $\rho(B)$ the resolvent set, $\sigma(B)$ the spectrum of B (that is, the complement of $\rho(B)$ in the complex plane) and we represent by $D(B)$ the domain of B. We consider $D(B)$ as a Banach space endowed with the graph norm. For $\lambda \in \rho(B)$, we denote by $R(\lambda, B) = (\lambda I - B)^{-1}$ the resolvent operator of B. For $r > 0$, the notation $B_r(X)$ stands for the closed ball $\{x \in X: \|x\| \leq r\}$.

2.1. S-asymptotically ω-periodic functions

The concept of S-asymptotically ω-periodicity for vector-valued functions is very recent. In last years some papers have studied properties and applications of these functions (see [3,14,17,18,29]). In the rest of this work $\omega > 0$ is a fixed real number.

Definition 2.1. A function $f \in C_b(\mathbb{R}^+; X)$ is said to be S-asymptotically ω-periodic if $\lim_{t\to\infty} (f(t+\omega) - f(t)) = 0$. In this case, we say that ω is an asymptotic period of f.

We use the notation $\mathrm{SAP}_\omega(X)$ (respectively, $\mathrm{AP}_\omega(X)$) to represent the subspace of $C_b(\mathbb{R}^+; X)$ formed by all functions S-asymptotically ω-periodic (respectively, asymptotically ω-periodic). We note that $\mathrm{SAP}_\omega(X)$ and $\mathrm{AP}_\omega(X)$ endowed with the norm of uniform convergence are Banach spaces. It is clear that every asymptotically ω-periodic function is S-asymptotically ω-periodic. However, the converse assertion is not verified. An important difference between both type of functions is that the range of an asymptotically ω-periodic function is relatively compact while the range of an S-asymptotically ω-periodic function is only a bounded set.

Definition 2.2. We say that a continuous function $f : \mathbb{R}^+ \times X \to Y$ is uniformly S-asymptotically ω-periodic on bounded sets of X if for every bounded subset K of X, the set $\{f(t, x): t \geq 0, x \in K\}$ is bounded and for every $\varepsilon > 0$ there is $T(K, \varepsilon) \geq 0$ such that $\|f(t, x) - f(t+\omega, x)\| \leq \varepsilon$ for all $t \geq T(K, \varepsilon)$ and all $x \in K$.

Definition 2.3. A continuous function $f : \mathbb{R}^+ \times X \to Y$ is said to be asymptotically uniformly continuous on bounded sets of X if for every $\varepsilon > 0$ and all bounded set $K \subseteq X$, there are constants $T = T_{\varepsilon,K}$ and $\delta = \delta_{\varepsilon,K} > 0$ such that $\|f(t, x) - f(t, y)\| \leq \varepsilon$, for all $t \geq T$ and $x, y \in K$ with $\|x - y\| < \delta$.

Lemma 2.4. *Assume that $f : \mathbb{R}^+ \times X \to Y$ is a function uniformly S-asymptotically ω-periodic on bounded sets of X and asymptotically uniformly continuous on bounded sets of X. Let $u : \mathbb{R}^+ \to X$ be an S-asymptotically ω-periodic function, then the Nemytskii function $F : \mathbb{R}^+ \to Y$ defined by $F(t) = f(t, u(t))$ is S-asymptotically ω-periodic.*

2.2. Pseudo S-asymptotically ω-periodic functions

We will review some results about pseudo S-asymptotically ω-periodic functions.