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Some quantum integral inequalities via preinvex functions

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ABSTRACT

In this paper, we obtain some new quantum analogues of Hermite–Hadamard and Iyengar type inequalities for some classes of preinvex functions. Some special cases are also discussed.

1. Introduction

It is well known that quantum calculus is the study of calculus without limits. Euler (1707–1783), was the first who started the study of quantum calculus. He introduced the q in tracks of Newton's infinite series. Formerly the study of quantum calculus started in early twentieth century with the work of F. H. Jackson. In quantum calculus, we obtain the q-analogues of mathematical objects which can be recaptured as $q \to 1$. It has been noticed that quantum calculus is subfield of time scale calculus. Time scale calculus provides a unified framework for studying dynamic equations on the both discrete and continuous domains. In quantum calculus, we are concerned with a specific time scale, called the q-time scale. The quantum calculus can be treated as bridge between Mathematics and Physics. Due significant applications of quantum calculus in Mathematics and Physics, this subject has received special attention by many researchers. As a result, quantum calculus has emerged as fascinating field. For some details on quantum calculus, see [9–11,24,26,27,33–35].

In recent years, theory of convexity has experienced rapid development. Many researchers extended and generalized the classical concepts of convex sets and convex functions in different directions, see [5,6,16]. A significant generalization of convex functions is invex function, which was introduced and studied by Hanson [12]. This work has greatly expanded the role of invexity in Optimization. Ben-Israel and Mond [2] introduced a class of functions, which is called preinvex functions as a generalization of convex functions. It is known that the differentiable preinvex functions are invex functions and the converse also holds under certain conditions, see [22]. Noor [17] has shown that the minimum of differentiable preinvex functions on the invex sets can be characterized by the variational-like inequalities. Variational-like inequalities can be viewed as a natural generalization of variational inequalities and related problems. In recent years, Pitea and Postolache [29–31] introduced the notion of quasi invexity and applied it to theoretical mechanics and nonlinear optimization. Later, Pitea and Antczak [32] introduced the concept of invexity and applied it to vector optimization. This shows that the preinvexity plays an important and significant role in the development of various fields of pure and applied sciences. For more details, see [1–3,15,17–22,25,29–32,36,37].

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An important factor which caused rapid development in theory of convexity is its close relationship with theory of inequalities. Many inequalities known in the literature are direct consequences of the applications of convex functions. An important inequality for convex functions which has been extensively studied in recent decades is Hermite–Hadamard's inequality, which was obtained by Hermite and Hadamard independently.

To be more precise, a function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function, where $a, b \in I$ with $a < b$, if and only if,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which is known as Hermite–Hadamard inequality.

The right side of Hermite–Hadamard inequality can be estimated by the inequality of Iyengar, which reads as:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \frac{M(b-a)}{4} - \frac{1}{4M(b-a)}(f(b)-f(a))^2,$$

where by M we denoted the Lipschitz constant, that is, $M = \sup\left\{\left|\frac{f(x)-f(y)}{x-y}\right|; x \neq y\right\}$.

This fundamental result of Hermite and Hadamard has attracted many mathematicians and consequently this inequality has been generalized and extended in different directions using novel and innovative ideas, see [4,6–8,13,14,16,18,20,20,21,23–26,28,33,34]. Noor [20,21] obtained Hermite–Hadamard's inequality for preinvex functions.

A function $f : I = [a + \eta(b, a)] \subset \mathbb{R} \to \mathbb{R}$ is a preinvex function with $\eta(b, a) > 0$, if and only if,

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Tariboon and Ntouyas [34] obtained q-analogue of Hermite–Hadamard's inequality as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\ {}_a\mathrm{d}_q x \leq \frac{qf(a)+f(b)}{1+q}. \tag{1.3}$$

Noor et al. [24,26] established some quantum estimates of Hermite–Hadamard and Iyengar type inequalities for some classes of convex functions.

In this paper, we obtain several quantum analogues of Hermite–Hadamard and Iyengar type inequalities for preinvex functions and prequasi-invex functions respectively. These quantum Hermite–Hadamard inequalities and their variant forms are useful for quantum physics where lower and upper bounds for natural phenomena described by integrals (such as mechanical work) are frequently required. Several special cases are also discussed. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

2. Preliminary results

In this section, we discuss some previously known concepts and results.

Let Ω be a nonempty closed set in $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ be a continuous function and let $\eta(\cdot,\cdot) : \Omega \times \Omega \to \mathbb{R}^n$ be a continuous bifunction.

Definition 2.1 ([12])**.** A set Ω is said to be invex set with respect to bifunction $\eta(\cdot,\cdot)$, if

$$u + t\eta(v, u) \in \Omega, \quad \forall u, v \in \Omega, t \in [0, 1]. \tag{2.1}$$

The invex set Ω is also called η-connected set.

Remark 1 ([1])**.** We would like to mention that Definition 2.1 of an invex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point u which is contained in Ω. We do not require that the point v should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that v should be an end point of the path for every pair of points $u, v \in \Omega$, then $\eta(v, u) = v - u$, and consequently invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(v, u) = v - u$, but the converse is not necessarily true, see [15,37] and the references therein.

Definition 2.2 ([2])**.** A function f is said to be preinvex with respect to arbitrary bifunction $\eta(\cdot,\cdot)$, if

$$f(u + t\eta(v, u)) \leq (1-t)f(u) + tf(v), \quad \forall u, v \in \Omega, t \in [0, 1]. \tag{2.2}$$

The function f is said to be preincave if and only if $-f$ is preinvex.

If $\eta(v, u) = v - u$, then preinvex functions become convex functions in the classical sense.

Definition 2.3. A function f on $\mathbb{R}$ is said to be convex in the classical sense, if

$$f(u + t(v-u)) \leq (1-t)f(u) + tf(v), \quad \forall u, v \in \Omega, t \in [0, 1].$$

From Definitions 2.3 and 2.4 it is obvious that every convex function is a preinvex function. However, it is known [36] that preinvex functions may not be convex functions.