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Simplified existence and uniqueness conditions for the zeros and the concavity of the *F* and *G* functions of improved Gauss orbit determination from two position vectors

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ABSTRACT

In our first paper we showed how Gauss's method for determining the initial position and velocity vectors from two inertial position vectors at two times in an idealized Keplerian two-body elliptical orbit can be made more robust and efficient by replacing functional iteration with Newton–Raphson iteration. To do this we split the orbit determination algorithm into two sub-algorithms, the x-iteration to find the zero of the fixed-point function $F(x)$ when the true anomaly angular difference between the two vectors is large and the y-iteration to find the zero of the fixed-point function $G(y)$ when the angular difference is small.

We derived necessity and sufficiency conditions for the existence and uniqueness of these zeros and conjectured that both functions were concave. In this paper we first simplify the necessity and sufficiency conditions. We then prove that both *F* and *G* are indeed concave.

Existence and uniqueness of the zero of *G* and its concavity then imply quadratic convergence of the Newton–Raphson iterative sequence from an arbitrary starting point in its domain. Quadratic convergence to the zero of $F(x)$ is also the case in its domain except for a small neighborhood of $x = 1$, corresponding to the true anomaly difference angle between the position vectors in a small neighborhood of 180°.

1. Introduction

In [1] we identified three major drawbacks of the classical Gauss method of determining idealized two-body elliptical orbital position and velocity from two position vectors and times via functional iteration. These were: (i) a relatively small range in the difference in the true anomaly angle between the two vectors for which the functional iteration converges. (ii) The need to have the initial guess $y_0 = 1$ be sufficiently close to the unknown fixed point of the functional iteration for convergence. (iii) Slow, linear convergence of the sequence of functional iterates if the method works at all.

We proposed a new approach in which the functional iteration is replaced by Newton–Raphson iteration on one of two fixed-point functions $F(x)$ or $G(y)$, the former being chosen if the true anomaly angular difference is large but less than π and the latter being chosen when the angular difference is small, say, greater than zero but less than or equal to $\pi/4$. Since our first publication Arroyo et al. in [2] constructed a class of fifth order derivative-free versions of our method for solving the Gauss problem. In a follow-on paper, co-written by some of the same authors of [2], and published in this journal [3], Andreu et al. have extended the construction to a class of efficient eighth order derivative-free versions of our method. All of these derivative-free methods,

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which solve for the zero of $F(x)$, not only retain the advantages of our Newton method but also obtain dramatically higher order convergence rates with improved numerical stability.

In [1] we derived necessary and sufficient inequality conditions, in terms of the characteristic parameters l and m, which guarantees existence and uniqueness of the solutions to the equations $F(x) = 0$ and $G(y) = 0$. These parameters are easily computed functions of the two position vectors and corresponding times.

To assess the robustness and efficiency of our approach we exercised its Matlab algorithm implementation on two test cases, a low eccentricity orbit from [5] and a high eccentricity Molniya orbit from [6] over a range of true anomaly angular differences. The results show that our approach resolved all three issues (i)–(iii), simultaneously. We were able to push the range of true anomaly difference out to over 177° for the low eccentricity orbit and over 176° for the Molniya orbit. Our method was insensitive to the starting guess, converging in all five sub-cases for each orbit. The algorithm convergence criterion was met in between four to eight iterations, depending on the size of the true anomaly difference, faster convergence for smaller angular differences, slower for larger angular difference. This last result was strongly suggestive of the quadratic convergence rate that is characteristic of Newton's method in general.

Finally, on the basis of the graphic evidence, we advanced the conjecture that both F and G are concave, i.e., $\frac{\partial^2 F}{\partial x^2} \le 0$ and $\frac{\partial^2 G}{\partial y^2} \le 0$ on their respective domains. It was noted that existence, uniqueness and concavity would then imply quadratic convergence independent of the starting point of the iterative sequence.

Our objective in this companion paper to [1] is to provide the proofs that underlie the computational properties of our method. These numerical characteristics are covered by way of example in the two orbital test cases described in [1] and, for the sake of brevity, are not covered again here.

The first result of this present, companion paper to [1] is a simple proof that $0 < l < \infty$. Then we show that m can be written as a cubic in l. Next we exploit the existence and uniqueness conditions derived in [1] to reduce them to a pair of simple inequalities. We show that the value of a positive parameter k, which is an easily computed function of the time difference between the two position vectors and the sum of their magnitudes, determines the necessity and sufficiency for existence and uniqueness. That is, $k \ge 2/9$ implies existence and uniqueness for all positive l. If $k < 2/9$ then existence and uniqueness hold if and only if $P(l) < 0$, P a monic cubic polynomial whose coefficients are linear functions of k.

The second, more important result is the proof of the concavity conjecture. Existence, uniqueness and concavity together then imply quadratic convergence to a unique fixed point from any starting point of the y-iteration and quadratic convergence for the x-iteration outside a small open interval containing $x = 1$. The concavity of F implies the insensitivity to initial guess for both my method and those described in [2] and [3]. Thus, the proof of F's concavity is an important result in itself.

The proof of the concavity of G is elementary, straightforward, and short. The proof for F is also elementary involving just High School algebra, trigonometry, and differential calculus. But it is trickier and required the development of a special approach for the proof of three trigonometric inequalities, each involving a quadratic polynomial with quadratic coefficients in the variable $U = 2\theta - \sin(2\theta), \theta \in [0, \pi)$.

The remainder of this paper is divided into five sections. Section 2 provides the notation and definitions that we will use throughout. Section 3 is divided into two parts. In the first part we derive simply evaluated necessary and sufficient conditions for existence and uniqueness of the zeros of F and G. The second part of the section is devoted to the proof of the concavity of F and G. Section 4 summarizes our conclusions and recommendations for future work. Section 5 identifies our references.

2. Notation and definitions

In what follows below we abstract the relevant notation from [1]:

μ	Earth's gravitational constant $(3.9860048 \times 10^{14}(m^3\text{sec}^{-2}))$
$\mathbf{x}_1$	the three-component orbit inertial position vector (m) at time t_1 (s)
$\mathbf{x}_2$	the three-component orbit inertial position vector (m) at time t_2 (s)
ν_1	true anomaly at $\mathbf{x}_1$
ν_2	true anomaly at $\mathbf{x}_2$
l	first Gaussian parameter
m	second Gaussian parameter
x	argument of F
y	argument of G

$$\tau = t_2 - t_1 > 0. \tag{1}$$

$$r_1 = (\mathbf{x}_1^T \mathbf{x}_1)^{1/2}. \tag{2}$$

$$r_2 = (\mathbf{x}_2^T \mathbf{x}_2)^{1/2}. \tag{3}$$

$$\mathbf{u}_1 = (1/r_1)\mathbf{x}_1. \tag{4}$$

$$\mathbf{u}_2 = (1/r_2)\mathbf{x}_2. \tag{5}$$