Applied Mathematics and Computation 269 (2015) 363–378

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Modified method of simplest equation for obtaining exact analytical solutions of nonlinear partial differential equations: further development of the methodology with applications

Nikolay K. Vitanov [a,b,*], Zlatinka I. Dimitrova [c], Kaloyan N. Vitanov [a]

[a] Institute of Mechanics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Bl. 4, 1113 Sofia, Bulgaria
[b] Max-Plank Institute for the Physics of Complex Systems, Nöthnitzer Str. 38, 01187 Dresden, Germany
[c] "G. Nadjakov" Institute of Solid State Physics, Tzarigradsko Chaussee Blvd. 72, 1784 Sofia, Bulgaria

ARTICLE INFO

Keywords:
Method of simplest equation
Nonlinear partial differential equations
Exact solutions

ABSTRACT

We discuss the application of a variant of the method of simplest equation for obtaining exact traveling wave solutions of a class of nonlinear partial differential equations containing polynomial nonlinearities. As simplest equation we use differential equation for a special function that contains as particular cases trigonometric and hyperbolic functions as well as the elliptic function of Weierstrass and Jacobi. We show that for this case the studied class of nonlinear partial differential equations can be reduced to a system of two equations containing polynomials of the unknown functions. This system may be further reduced to a system of nonlinear algebraic equations for the parameters of the solved equation and parameters of the solution. Any nontrivial solution of the last system leads to a traveling wave solution of the solved nonlinear partial differential equation. The methodology is illustrated by obtaining solitary wave solutions for the generalized Korteweg–deVries equation and by obtaining solutions of the higher order Korteweg–deVries equation.

1. Introduction

Traveling wave solutions of nonlinear partial differential equations are studied much in the last decades [1–5] as they occur in many natural systems [6–9] and because of existence of various methods for obtaining such solutions [10–14]. Below we shall consider the method of simplest equation for obtaining exact analytical solutions of nonlinear partial differential equations [15–18] and especially its version called modified method of simplest equation [19–21]. Method of simplest equation is based on a procedure analogous to the first step of the test for the Painleve property [18,22]. In the version of the method called modified method of the simplest equation [23,24] this procedure is substituted by the concept for the balance equation. Modified method of simplest equation has its roots back in the history (for an example see [25–28]). Method of simplest equation has been successfully applied for obtaining exact traveling wave solutions of numerous nonlinear PDEs such as versions of generalized Kuramoto–Sivashinsky equation, reaction– diffusion equation, reaction - telegraph equation, generalized Swift – Hohenberg equation and generalized Rayleigh equation, generalized Fisher equation, generalized Huxley equation , generalized Degasperis – Procesi equation and b-equation, extended Korteweg–de Vries equation, etc. [29–35].

* Corresponding author. Tel.: +35929796416.
E-mail address: vitanov@imbm.bas.bg, n.k.vitanov@gmail.com (N.K. Vitanov).

http://dx.doi.org/10.1016/j.amc.2015.07.060

A short summary of the method of simplest equation is as follows. First of all by means of an appropriate ansatz (for an example the traveling-wave ansatz) the solved nonlinear partial differential equation is reduced to a nonlinear ordinary differential equation

$$P\Big(u, u_\xi, u_{\xi\xi}, \dots\Big) = 0 \tag{1.1}$$

Then the finite-series solution

$$u(\xi) = \sum_{\mu=-\nu}^{\nu_1} p_\mu [g(\xi)]^\mu \tag{1.2}$$

is substituted in (1.1). p_μ are coefficients and $g(\xi)$ is solution of simpler ordinary differential equation called simplest equation. Let the result of this substitution be a polynomial of $g(\xi)$. Eq. (1.2) is a solution of Eq. (1.1) if all coefficients of the obtained polynomial of $g(\xi)$ are equal to 0. This condition leads to a system of nonlinear algebraic equations. Each solution of the last system leads to a solution of the studied nonlinear partial differential equation.

In this article we consider a large class of (1+1)-dimensional nonlinear partial differential equations that are constricted by polynomials of the unknown function and its derivatives. As simplest equation we shall use equation of the kind

$$\left(\frac{dg}{d\xi}\right)^2 = \sum_{i=0}^{m} a_i g^i.$$

The text below is organized as follows. In Section 2 we introduce the class of studied nonlinear partial differential equations and the used class of simplest equations and their solutions. Then we show that any of the nonlinear partial differential equations of the discussed class can be reduced to a system of two equations containing polynomials of the unknown function. These polynomials can be obtained on the basis of addition and multiplication of some basic polynomials connected to the derivatives of the solved nonlinear partial differential equation. In Section 3 we calculate some of the most used basic polynomials. In Section 4 the methodology is illustrated by application for obtaining solitary wave solutions of

- Generalized Korteweg–deVries equation;
- Higher order Korteweg–deVries equation.

Several concluding remarks are summarized in Section 5.

2. Formulation of the method

2.1. Proof of the basic theorem

Let us consider a nonlinear PDE with nonlinearities that are polynomials of the unknown function $h(x, t)$ and its derivatives. We search solution of the kind

$$h(x,t) = h(\xi); \quad \xi = \mu x + \nu t \tag{2.1}$$

where μ and ν are parameters. The basis of our search will be a solution $g(\xi)$ of a certain simplest equation. Hence

$$h = f[g(\xi)] \tag{2.2}$$

h from Eq. (2.2) is a composite function. For the nth derivative of h we have the Faa di Bruno formula [36]

$$h_{(n)} = \sum_{k=1}^{n} f_{(k)} \sum_{p(k,n)} n! \prod_{i=1}^{n} \frac{g_{(i)}^{\lambda_i}}{(\lambda_i!)(i!)^{\lambda_i}} \tag{2.3}$$

where

- $h_{(n)} = \frac{d^n h}{dx^n}$;
- $f_{(k)} = \frac{d^k f}{dg^k}$;
- $g_{(i)} = \frac{d^i g}{dx^i}$;
- $p(n,k) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$: set of numbers such that

$$\sum_{i=1}^{n} \lambda_i = k; \sum_{i=1}^{n} i\lambda_i = n. \tag{2.4}$$

Further we shall concentrate on $f_{(k)}$ and $g_{(i)}$.

Let us now assume that f is a polynomial of g. Then

$$f = \sum_{r=0}^{q} b_r g^r \tag{2.5}$$