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Analysis of convergence for the alternating direction method applied to joint sparse recovery ☆

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ABSTRACT

The sparse representation of a multiple measurement vector (MMV) is an important problem in compressed sensing theory, the old alternating direction method (ADM) is an optimization algorithm that has recently become very popular due to its capabilities to solve large-scale or distributed problems. The MMV–ADM algorithm to solve the MMV problem by ADM has been proposed by H. Lu, et al. (2011)[24], but the theoretical result about the convergence of matrix iteration sequence generated by the algorithm is left as a future research topic. In this paper, based on the subdifferential property of the two-norm for vector, a shrink operator associated with matrix is established. By using the operator, a convergence theorem is proved, which shows the MMV–ADM algorithm can recover the jointly sparse vectors.

1. Introduction

Compressed sensing has been a very active field of recent research with a wide range of applications, including signal processing [1,2], medical imaging [3], radar system [4,5], and image compression [6,7]. A central goal is to develop fast algorithms that can recover sparse signals from a relatively small number of linear measurements. The single measurement vector (SMV) formulation is now standard in sparse approximation and compressed sensing literature. For a signal $x \in R^n$, define $\|x\|_0$ to be the number of nonzero elements of x. Now we want to recover the original signal x from a linear measurement $b = Ax$, where A is a known $m \times n$ matrix ($m \ll n$). Then we need to solve the problem

$$\min \|x\|_0 \quad s.t. \quad Ax = b, \tag{1.1}$$

where A and b are known.

However, it is well-known that these problems are NP-hard and convex relaxations of these problems have been proposed and studied in the literature. Candès and Tao [8] introduced an ℓ_1 minimization method for the sparse signal recovery,

$$\min \|x\|_1 \quad s.t. \quad Ax = b, \tag{1.2}$$

where $\|x\|_1 = \sum_{i=1}^n |x_i|$ is the standard ℓ_1 norm. The study of this problem (1.2) was pioneered by Donoho, Candès, and their collaborators [2,8–10]. Many researchers have made a lot of contributions related to the existence, uniqueness, and other properties of the sparse solution as well as computational algorithms and their convergence analysis to tackle problem (1.1) in [11,12].

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A natural extension of the problem (1.1) is the joint sparse recovery problem, also known as the multiple measurement vector (MMV) problem, aims to identify a common support shared by unknown sparse vectors $x_1, \ldots, x_n$ from the multiple vectors $b_k = Ax_k$ for $k = 1, .., n$ obtained through a common sensing matrix A. Several MMV recovery methods were proposed in recent years. Most of the methods are extensions of existing SMV recovery methods. The MMV problem is considered in [13,14], where the objective is to minimize the number of rows containing nonzero entries. Given a multiple-measurement vector $B \in R^{m\times L}$ and a dictionary $A \in R^{m\times n}$, the MMV problem can be formulated as

$$\min \|X\|_{row} \quad s.t. \quad AX = B, \tag{1.3}$$

where $\|X\|_{row}$ is defined as the number of nonzero rows of the matrix X. A sparse representation means that matrix X (or a vector, if one has an SMV: $L = 1$) has a small number of rows that contain nonzero entries. This problem (1.3) can be formulated as follows

$$\min \|X\|_{2,0} \quad s.t. \quad AX = B, \tag{1.4}$$

where the (r, p)-norm of X is defined as:

$$\|X\|_{r,p} = \left(\sum_{i=1}^{n} \|X_i\|_r^p\right)^{1/p}, 1 \le r \le \infty, \ p > 0$$

and X_i denotes the ith row of X. Moreover, the $(r, 0)$-norm of X is defined as:

$$\|X\|_{r,0} = \lim_{p\to 0_+} \|X\|_{r,p}^p, \ 1 \le r \le \infty.$$

However, (1.4) is a combinatorial optimization problem and is thus NP-hard. Similar to the use of ℓ_1-norm to replace ℓ_0-norm in SMV, the matrix (2,0)-norm in (1.4) is usually replaced by the matrix (2,1)-norm that results in the following convex relaxation problem

$$\min \|X\|_{2,1} \quad s.t. \quad AX = B. \tag{1.5}$$

Minimizing the (2,1)-norm in (1.5) plays a central role in promoting solution sparsity, it is easy to see that the matrix (2,1)-norm is the ℓ_1 norm of the vector consisting of the ℓ_2 norm of rows of X as its components. Problem (1.5) shares common solutions with (1.4) under some favorable conditions (see, [15]). When B contains noise, we consider the following constrained optimization problem

$$\min \|X\|_{2,1} \quad s.t. \quad \|AX - B\|_F \le \delta. \tag{1.6}$$

From optimization theory, it is easy to see that problems (1.5) or (1.6) can be transformed into the following unconstrained problem

$$\min_X \|X\|_{2,1} + \frac{1}{2\mu}\|AX - B\|_F^2. \tag{1.7}$$

In the last few years, numerous algorithms have been proposed and studied for solving the MMV problem. Orthogonal matching pursuit (OMP) algorithms [13,14,16,17] are extended to the MMV case, and convex optimization formulations with mixed norm [13,14,18] extend the corresponding SMV solution, such as basis pursuit (BP) [19] and LASSO [20] to the MMV case. Sparse bayesian learning (SBL) [21] has also been extended to the MMV case [22]. Theories have also been developed for both the joint sparse recovery problem itself and the guarantees of the algorithms that solve it in [13–15,23]. The MMV–ADM algorithm to solve the problem (1.7) has been proposed in [24], but the theoretical result about the convergence of matrix iteration sequence is left as a future research topic. In this paper, based on the subdifferential property of the two-norm for vector, a shrink operator associated with matrix is established. By using the operator, a convergence theorem is proved, which shows the MMV–ADM algorithm can solve the problem (1.7).

The rest of the paper is organized as follows. In Section 2, we shall introduce some notations and preliminary lemmas. In Section 3, we shall give analysis of convergence for the alternating direction method to solve the problem (1.7). In Section 4, Numerical results are presented and demonstrate a notable ability to quickly decrease the relative error to true solutions. Finally, Section 5 concludes the paper.

2. Notations and preliminaries

In this section, we introduce basic notations and preliminary lemmas that will be used throughout the paper.

For matrices A and B of the same dimensions, we define the inner product in $R^{m\times n}$ as $\langle A, B\rangle = tr(A^TB) = \sum_{i=1}^m \sum_{j=1}^n A_{ij}B_{ij}$. The norm associated with this inner product is called the Frobenius (or Hilbert–Schmidt) norm $\|\cdot\|_F$. If $M \in R^{m\times m} \succ 0$ is a symmetric positive definite matrix, we let $\langle X, MX\rangle = \|X\|_M^2$. The norm $\|\cdot\|$ refers to the two-norm (spectral norm) of vector (matrix).

The following lemmas are necessary for joint sparse vectors recovery.

Lemma 2.1. *([25]) For $x \in R^n$ and $A \in R^{m\times n}$, the subdifferential of $f(x) \triangleq \|Ax\|$ is*