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Degenerate poly-Cauchy polynomials

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ABSTRACT

In this paper, we study several properties for the degenerate poly-Cauchy polynomials. We present several explicit formulas and recurrence relations for these polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.

1. Introduction

The *poly-Cauchy polynomials* of the first kind $C_n^{(k)}(x)$ and of the second kind $\widehat{C}^{(k)}(x)$ are respectively defined by

$$Lif_k(\log(1+t))(1+t)^x = \sum_{n\ge 0} C_n^{(k)}(x)\frac{t^n}{n!},$$

$$Lif_k(-\log(1+t))(1+t)^{-x} = \sum_{n\ge 0} \widehat{C}_n^{(k)}(x)\frac{t^n}{n!},$$

for all $k \in \mathbb{Z}$, where

$$Lif_k(x) = \sum_{m\ge 0} \frac{x^m}{m!(m+1)^k} \tag{1.1}$$

is the *polylogarithm factorial function*. When $x = 0$, $C_n^{(k)} = C_n^{(k)}(0)$ and $\widehat{C}_n^{(k)} = \widehat{C}_n^{(k)}(0)$ are respectively called the *poly-Cauchy numbers* of the first kind and of the second kind. The poly-Cauchy polynomials, poly-Cauchy numbers, Barnes-type Daehee of the first kind and poly-Cauchy of the first kind mixed-type polynomials, and the polylogarithm factorial function are found in [12–14].

Here we introduce the degenerate versions of the poly-Cauchy polynomials. Namely, the *degenerate poly-Cauchy polynomials* $C_n^{(k)}(\lambda, x)$ *of the first kind and* $\widehat{C}_n^{(k)}(\lambda, x)$ *of the second kind* are respectively given by

$$Lif_k\left(\frac{(1+t)^\lambda - 1}{\lambda}\right)(1+t)^x = \sum_{n\ge 0} C_n^{(k)}(\lambda, x)\frac{t^n}{n!}, \tag{1.2}$$

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$$Lif_k\left(-\frac{(1+t)^\lambda-1}{\lambda}\right)(1+t)^{-x}=\sum_{n\geq 0}\widehat{C}_n^{(k)}(\lambda,x)\frac{t^n}{n!}. \tag{1.3}$$

We observe that $\lim_{\lambda\to 0}C_n^{(k)}(\lambda,x)=C_n^{(k)}(x)$ and $\lim_{\lambda\to 0}\widehat{C}_n^{(k)}(\lambda,x)=\widehat{C}_n^{(k)}(x)$. When $x=0$, $C_n^{(k)}(\lambda,0)$ and $\widehat{C}_n^{(k)}(\lambda,0)$ are respectively called the *degenerate poly-Cauchy numbers* of the first kind and of the second kind.

The purpose of this paper is to use umbral calculus techniques (see [17,18]) in order to derive some properties, recurrence relations, and identities for the degenerate poly-Cauchy polynomials of the first kind and of the second kind. From (1.2) and (1.3), it is immediate to see that $C_n^{(k)}(\lambda,x)$ is the Sheffer sequence for the pair $g(t)=\frac{1}{Lif_k(\frac{e^{\lambda t}-1}{\lambda})}$, $f(t)=e^t-1$, and that $\widehat{C}_n^{(k)}(\lambda,x)$ is the Sheffer sequence for the pair $g(t)=\frac{1}{Lif_k(-\frac{e^{-\lambda t}-1}{\lambda})}$, $f(t)=e^{-t}-1$. Thus,

$$C_n^{(k)}(\lambda,x)\sim\left(\frac{1}{Lif_k\left(\frac{e^{\lambda t}-1}{\lambda}\right)},e^t-1\right), \tag{1.4}$$

$$\widehat{C}_n^{(k)}(\lambda,x)\sim\left(\frac{1}{Lif_k\left(-\frac{e^{-\lambda t}-1}{\lambda}\right)},e^{-t}-1\right). \tag{1.5}$$

Umbral calculus has been used in numerous problems of mathematics (for example, see [1,3,5,7,8]) and used in different areas of physics; for example it is used in group theory and quantum mechanics by Biedenharn et al. [5,6] (for other examples, see [9,11–14,16,19] and references therein).

2. Explicit expressions

In this section we present several explicit formulas for the degenerate poly-Cauchy polynomials, namely $C_n^{(k)}(\lambda,x)$ and $\widehat{C}_n^{(k)}(\lambda,x)$. To do so, we recall that Stirling numbers $S_1(n,k)$ of the first kind can be defined by means of exponential generating functions as

$$\sum_{\ell\geq j}S_1(\ell,j)\frac{t^\ell}{\ell}=\frac{1}{j!}\log^j(1+t) \tag{2.1}$$

and can be defined by means of ordinary generating functions as

$$(x)_n=\sum_{m=0}^{n}S_1(n,m)x^m\sim(1,e^t-1), \tag{2.2}$$

where $(x)_n=x(x-1)(x-2)\cdots(x-n+1)$ with $(x)_0=1$.

Theorem 2.1. *For all* $n\geq 0$,

$$C_n^{(k)}(\lambda,x)=\sum_{j=0}^{n}\left(\sum_{\ell=j}^{n}\binom{n}{\ell}S_1(\ell,j)C_{n-\ell}^{(k)}(\lambda,0)\right)x^j,$$

$$\widehat{C}_n^{(k)}(\lambda,x)=\sum_{j=0}^{n}(-1)^j\left(\sum_{\ell=j}^{n}\binom{n}{\ell}S_1(\ell,j)\widehat{C}_{n-\ell}^{(k)}(\lambda,0)\right)x^j,$$

Proof. It is well known that $s_n(x)=\sum_{j=0}^{n}\frac{1}{j!}\langle\frac{1}{g(\bar f(t))}\bar f(t)^j|x^n\rangle x^j$, for all $s_n(x)\sim(g(t),f(t))$ (see [17,18]). For the pair $(g(t),f(t))$ in (1.4), we obtain

$$\begin{aligned}\left\langle\frac{1}{g(\bar f(t))}\bar f(t)^j|x^n\right\rangle&=\left\langle Lif_k\left(\frac{(1+t)^\lambda-1}{\lambda}\right)(\log(1+t))^j|x^n\right\rangle\\&=\left\langle Lif_k\left(\frac{(1+t)^\lambda-1}{\lambda}\right)|(\log(1+t))^jx^n\right\rangle.\end{aligned}$$

Thus, by (2.1) and then by (1.2), we have

$$\begin{aligned}\left\langle\frac{1}{g(\bar f(t))}\bar f(t)^j|x^n\right\rangle&=j!\sum_{\ell=j}^{n}\binom{n}{\ell}S_1(\ell,j)\left\langle Lif_k\left(\frac{(1+t)^\lambda-1}{\lambda}\right)|x^{n-\ell}\right\rangle\\&=j!\sum_{\ell=j}^{n}\binom{n}{\ell}S_1(\ell,j)C_{n-\ell}^{(k)}(\lambda,0),\end{aligned}$$

which implies that $C_n^{(k)}(\lambda,x)=\sum_{j=0}^{n}(\sum_{\ell=j}^{n}\binom{n}{\ell}S_1(\ell,j)C_{n-\ell}^{(k)}(\lambda,0))x^j$. Similarly, for the pair $(g(t),f(t))$ in (1.5), we obtain the formula for the poly-Cauchy polynomials of the second kind. □