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# Method of weighted expected residual for solving stochastic variational inequality problems

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A R T I C L E I N F O

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A B S T R A C T

This paper is concerned in constructing a deterministic model for the stochastic affine variational inequality problems with nonlinear perturbation (for short, SVIPP) based on the convex combined expectations of the least absolute deviation and least squares about the so-called regularized gap function. We formulate SVIPP as a weighted expected residual minimization problem (in short, WERM). Some properties of the WERM problem are derived under suitable conditions. Moreover, we obtain a discrete approximation of WERM problem by applying the quasi-Monte Carlo method. The limiting behavior of optimal solutions and stationary points of the approximation problem are analyzed as well.



## 1. Introduction

Variational inequality is a unifying theme for the study of optimization and equilibrium problems, and it plays as an effective framework in numerical algorithms of optimization problems; see [3,5,12] and the references therein. A large number of real life applications, such as transportation, economics, engineering and so on, can be formulated as variational inequality problems. As in many practical problems, some elements may involve uncertain data, stochastic variational inequality problem has been receiving tremendous attention. Therefore it is necessary and meaningful to study the stochastic variational inequality problems.

Stochastic variational inequality problems may has no solution because the number of random events can be very large in practice. Therefore, it is significant to construct a deterministic model so as to provide a reasonable resolution for the stochastic variational inequality problems. Recently, there are many papers devoted to studying the stochastic variational inequality problems; see [1,2,4,7,10,11,13–16,21–23] and the references therein. In general, three major approaches, which include the expectation approach, the robust programming approach and the chance programming approach, are considered to handle random data; see [20] for more details. We are concerned in this paper with the expectation approach, which may fit the center of location of the data well.

Recently, Luo and Lin [13] formulated the stochastic affine variational inequality problems as an optimization problem (ERM problem) that minimizes the expected residual of the so-called regularized gap function. Properties of the ERM problem were presented and a quasi-Monte Carlo method was applied for solving the problem, along with its convergence analysis. Moreover, for closed sample space, Luo and Lin [14] presented a compact approximation approach for a class of stochastic variational inequality problems, in which the involved function is nonlinear. Based on [13,14], Ma et al. [16] formulated the stochastic affine variational inequality problems with nonlinear perturbation as an ERM problem, and studied some properties of the ERM problem under suitable conditions. In addition, they obtained a discrete approximation of ERM problem by means of quasi-Monte

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Carlo method. The convergence of optimal solutions and stationary points of the approximation problem was also studied with sample size increasing. It is worth pointing out that all the mentioned ERM problems can be seen as the least absolute deviation ( in short, LAD) loss in statistics.

Although LAD regression method is a robust statistical model tool in the case of data sets subjected to heavy-tailed errors or outliers, it has some limitations in terms of uniqueness and efficiency of the solution. Specifically, since the loss function for LAD is not strictly convex, which may lead to the solution not necessarily be unique in general. Moreover, when the noise follows a normal distribution, that is to say there exist few extreme samples in the data sets, the efficiency of LAD will be reduced in this case compared with the well-known ordinary least squares (LS) regression, which is the most convenient and efficient approach. However, the LS method is sensitive to the outliers data distribution, which indicates that it is not robust, see [6,19] for more details. Thus, it is desirable to develop a new modeling procedure in order to achieve both robustness and efficiency by adapting to different types of sample spaces. Motivated by these observations, in this paper, for stochastic affine variational inequality problems with a nonlinear perturbation, we propose a robust and efficient version based on the convex combined expectations of LAD and LS about the regularized gap function, named as the WERM problem. The differentiability of the objective function as well as the error bound on solution of WERM problem are characterized, we further obtain that the level set of the objective function is bounded. Moreover, by means of quasi-Monte Carlo method, we present a discrete approximation of WERM problem. Finally, we show that the sequences of optimal solutions and stationary points of the approximation problem converges to the true optimal value and true stationary point as the sample size goes to infinity, respectively.

The rest of this paper is organized as follows. In Section 2, we recall some notions and preliminary results. In Section 3, some properties of the WERM problem are obtained under certain suitable conditions, and a discrete approximation of WERM problem is followed by using the quasi-Monte Carlo method. Then, the limiting behavior of optimal solutions and stationary points of the approximation problem are studied in Section 4.

## 2. Preliminaries

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space. The classical variational inequality problem, denoted by *VI*($f$, $S$), is to find a vector $\acute{x} \in S$ such that

$$(x-\acute{x})^\top f(\acute{x}) \geq 0, \ \ \forall x \in S,$$

where $S \subseteq \mathbb{R}^n$ is a nonempty closed convex set and $f : \mathbb{R}^n \to \mathbb{R}^n$. In [8,9], Fukushima defined a regularized gap function

$$g(x) := \max_{y\in S}\{(x-y)^\top f(x) - \frac{\alpha}{2}\|x-y\|_G^2\},$$

where $\alpha$ is a positive parameter, $G$ is an $n \times n$ symmetric positive-definite matrix, and $\|\cdot\|_G$ means the $G-$norm defined by $\|x\|_G = \sqrt{x^\top Gx}$ for $x \in \mathbb{R}^n$. It has been shown that

(1) $g(x) \geq 0$ for every $x \in S$;
(2) For $x \in S, g(x) = 0$ if and only if $x$ solves *VI*($f$, $S$);
(3) $H(x) := Proj_{S,G}(x - \alpha^{-1}G^{-1}f(x))$ is the unique solution of the problem:

$$\max_{y\in S}\{(x-y)^\top f(x) - \frac{\alpha}{2}\|x-y\|_G^2\},$$

where $Proj_{S,\,G}$ denotes the projection operator onto $S$ under the $G-$norm.

Clearly, *VI*($f$, $S$) is equivalent to the following minimization problem:

$$\min_{x\in S} g(x),$$

and if $f$ is continuously differentiable, then $\nabla g(x) = f(x) - (\nabla f(x) - \alpha G)(H(x) - x)$.

Recently, stochastic variational inequality problem has attracted considerable attention, which is to find a vector $\tilde{x} \in S$ such that

$$(x-\tilde{x})^\top F(\tilde{x}, \omega) \geq 0, \ \ x \in S, \omega \in \Omega \, a.s. \tag{1}$$

where $F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ is a mapping, $\Omega \subset \mathbb{R}^m$ is the underlying sample space and a.s. is short for “almost surely” under the given probability measure. We choose a scalar $\alpha > 0$ and an $n \times n$ symmetric positive definite matrix $G$. Following [8,9], we define a regularized gap function $g : \mathbb{R}^n \times \Omega \to [0, \infty)$ for (1) as follows:

$$g(x, \omega) := \max_{y\in S}\{(x-y)^\top F(x, \omega) - \frac{\alpha}{2}\|x-y\|_G^2\}. \tag{2}$$

In view of Theorem 10.2.3 (a) of [5], for any $x \in S$ and $\omega \in \Omega$, it follows that

$$g(x, \omega) = (x - H(x, \omega))^\top F(x, \omega) - \frac{\alpha}{2}\|x - H(x, \omega)\|_G^2, \tag{3}$$

where

$$H(x, \omega) := proj_{S,G}(x - \alpha^{-1}G^{-1}F(x, \omega)). \tag{4}$$

In the following, we suppose that $g(x, \cdot)$ is integrable on $\Omega$ for each $x \in S$, and $n \times n$ symmetric positive-definite matrix G is given.