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On backwards and forwards reachable sets bounding for perturbed time-delay systems

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A R T I C L E I N F O

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A B S T R A C T

Linear systems with interval time-varying delay and unknown-but-bounded disturbances are considered in this paper. We study the problem of finding outer bound of forwards reachable sets and inter bound of backwards reachable sets of the system. Firstly, two definitions on forwards and backwards reachable sets, where initial state vectors are not necessary to be equal to zero, are introduced. Then, by using the Lyapunov–Krasovskii method, two sufficient conditions for the existence of: (i) the smallest possible outer bound of forwards reachable sets; and (ii) the largest possible inter bound of backwards reachable sets, are derived. These conditions are presented in terms of linear matrix inequalities with two parameters need to tuned, which therefore can be efficiently solved by combining existing convex optimization algorithms with a two-dimensional search method to obtain optimal bounds. Lastly, the obtained results are illustrated by four numerical examples.

1. Introduction

Within recent years, there has been an increasing interest in the problem of reachable set bounding for time-delay systems perturbed by unknown-but-bounded disturbances [1–15]. Most of the existing results on this topic have been based on the definition introduced in [16] that "Reachable set of dynamic systems perturbed by bounded inputs (disturbances) is the set of all the states that are reachable from the origin, in finite time, by inputs with peak value".

In many practical applications, for example, in safety verification, model checking [17–20], one usually requires to find a bound of a set of all the states that are reachable from a given set (not only from the origin point). A general definition on reachable set (forwards reachable set) to express a requirement like this, and a converse definition on backwards reachable set, have been given for perturbed systems without time-delay [17]. These notions have been widely applied to safety verification, model checking, state bounding observers, etc. (see [17–20] and the references therein). Forwards reachable set with respect to a given initial set of a perturbed dynamic system is the set of all the states starting from this given initial set. Backwards reachable set with respect to a given set in state vector space (called a target set) is the set of initial state such that the set of all the states starting from this initial set is covered by the given target set. It is easy to see that reachable set defined in [16] is a special case of forwards reachable set when the given initial set contains only the origin point. Most of existing results on forwards and backwards reachable sets bounding reported on perturbed systems without time-delay and, to our knowledge, there has not been any research attention reported on an extension to perturbed time-delay systems. Motivated by this, in this paper, we study

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a new problem of finding the smallest possible outer bound of forwards reachable sets with respect to a given set of initial state for perturbed time-delay systems and a converse problem of finding the largest possible inner bound of backwards reachable sets with respect to a given target set.

With regard to the problem of reachable set bounding for perturbed linear time-delay systems, whose matrices are constant, the widely used approach is based on the Lyapunov method and linear matrix inequalities [1–13,16]. Based on the Lyapunov Razumikhin method, Fridman and Shaked [16] first reported a result on reachable set bounding for perturbed linear time-delay systems. Later, based on modified Lyapunov–Krasovskii functionals, improved and extended results were reported for linear systems [1,3,5–7,11], neutral systems [2], systems with distributed delay [4,10], discrete-time systems [8,9,13], neural networks systems [12]. It is worthwhile to note that there is another approach based on linear positive systems and was proposed recently for linear time-varying systems [14] or nonlinear systems [15]. In this paper, we also use the Lyapunov–Krasovskii method to study the problem of forwards and backwards reachable set bounding for perturbed linear time-delay systems. Firstly, we propose an extended Lyapunov–Krasovskii functional (LKF) in which a delay-dependent matrix is incorporated. Note that the time-delay dependent matrix technique is first introduced in [21]. This technique allows one to exploit information of the upper and lower bound of derivative of time-varying delay and to reduce requirement of the existence of common matrix variables. Hence, the condition obtained by using this technique is expected to be less conservative. Next, we use the Wirtinger-based integral inequality [22,23] and the reciprocally convex technique [24] in estimating the derivative of the proposed LKF. The Wirtinger-based inequality, which encompasses the well-known Jensen inequality was introduced in [22,23] and further developed in [25–29]. Also note that this inequality in combination with the reciprocally convex technique [24] gives a more effective estimation of the derivative of the proposed LKF. As a result, we obtain two delay-derivative-dependent sufficient conditions for the existence of the smallest possible ball which outer bounds forwards reachable sets and the largest possible ball which inter bounds backwards reachable sets of the system. These new conditions are given in terms of linear matrix inequalities with two parameters need to tuned, which therefore can be efficiently solved by using existing convex optimization algorithms combining with a two-dimensional search method and allow us to obtain optimal possible bounds. To further optimize the obtained bounds, the technique on optimization on each axis with different exponential rates [7,8] is also used in the derivation of our results. Lastly, the feasibility and the effectiveness of the obtained results are illustrated by four numerical examples.

This paper is organized as follows. After the introduction, the problem statement and preliminaries are introduced in Section 2. The main results are given in Section 3. Four numerical examples are given in Section 4. Finally, a conclusion is drawn in Section 5.

2. Problem statement and preliminaries

Consider the following system

$$\begin{aligned} &\dot{x}(t) = Ax(t) + A_1x(t-\tau(t)) + B\omega(t), \quad t \geq 0, \\ &x(s) \equiv \varphi(s), \quad s \in [-\tau_M, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$, $A_1 \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ are known constant matrices. The time-varying delay is assumed to be differentiable and satisfying

$$0 \leq \tau_m \leq \tau(t) \leq \tau_M, \quad d_m \leq \dot{\tau}(t) \leq d_M \leq 1. \tag{2}$$

The function $\varphi(s) \in C_1([-\tau_M, 0], \mathbb{R}^n)$ is the initial condition function satisfying

$$\sup_{s\in[-\tau_M,0]} \dot{\varphi}^T(s)\dot{\varphi}(s) \leq \mu^2. \tag{3}$$

The disturbance vector, $\omega(t) \in \mathbb{R}^p$, is unknown but it is assumed to be bounded

$$\omega^T(t)\omega(t) \leq \omega^2, \quad \forall t \geq 0. \tag{4}$$

Here, τ_m, τ_M, μ, ω are given non-negative scalars, d_m and d_M are given scalars.

Definition 1 (Forwards and backwards reachable sets)**.** (i) Given a closed convex set, which contains the origin point, $\Omega_0 \in \mathbb{R}^n$ (called an initial set). A set $\Omega \in \mathbb{R}^n$ is called *a forwards reachable set with respect to the given initial set* Ω_0 of system (1) with conditions (2), (3) and (4) if for all initial condition function $\varphi(s) \in \Omega_0, \forall s \in [-\tau_M, 0]$, the solution $x(t, \varphi, \omega(t)) \in \Omega, \forall t \geq 0$.

(ii) Given a closed convex set, which contains the origin point, $\Lambda \in \mathbb{R}^n$ (called a target set). A set $\Lambda_0 \in \mathbb{R}^n$ is called *a backwards reachable set with respect to the given target set* Λ of system (1) with conditions (2), (3) and (4) if for all initial condition function $\varphi(s) \in \Lambda_0, \forall s \in [-\tau_M, 0]$, the solution $x(t, \varphi, \omega(t)) \in \Lambda, \forall t \geq 0$.

The objective of this paper is to find the smallest possible outer bound of forwards reachable sets with respect to a given initial set; and the largest possible inner bound of backwards reachable sets with respect to a given target set.

The following lemmas are useful for our main results.

Lemma 1. *For a given positive scalar* α*, let* V *be a Lyapunov–Krasovskii-like function for system (1)–(4). If* $\dot{V}(t) + \alpha V(t) - \frac{\alpha}{\omega^2}\omega^T(t)\omega(t) \leq 0, \forall t \geq 0$ *then*

$$V(t) \leq \max\{1, V(0)\}, \quad \forall t \geq 0.$$