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A generalization of preconditioned parameterized inexact Uzawa method for indefinite saddle point problems☆

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ABSTRACT

The parameterized inexact Uzawa methods have been used to solve some of the symmetric saddle point problems. In this paper, a new preconditioned parameterized inexact Uzawa method is presented to solve indefinite saddle point problems. After preconditioning, theoretical analyses show that the iteration method converges under certain conditions. So we propose three new algorithms based on these conditions. Numerical experiments are provided to show the effectiveness of the proposed preconditioner and all these algorithms have fantastic convergence rates by choosing optimal parameters.

1. Introduction

We consider the solution of systems of linear equations of the 2×2 block form

$$\mathcal{A}u = \begin{bmatrix} A & B^T \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} = b, \tag{1.1}$$

where $A \in R^{n\times n}$ is nonsingular, $B, C \in R^{m\times n}$ $(m \le n)$ are of full row rank, $x, f \in R^n$ and $y, g \in R^m$, B^T denotes the transpose of the matrix B.

The linear system (1.1) is called a generalized saddle point problem. The saddle point problems arise in a wide variety of technical and scientific applications, such as constrained and weighted least squares estimation, constrained optimization, computational fluid dynamics and mixed finite element approximations of elliptic PDEs and so on, see [1,2]. Moreover, the solutions of these problems have been stated and classified in detail [2].

A large amount of works have been devoted to developing efficient methods for the linear system (1.1) in recent years. In order to improve the convergent rates of iterative methods, it is often advantageous to employ a preconditioner. In particular, when $\mathcal{A}$ is nonsymmetric, Krylov subspace methods are often applied such as GMRES [3]; however, such methods tend to converge slowly. Furthermore, preconditioners can be constructed from matrix splitting iterative methods or matrix factorization. It can also be constructed by special structure of the coefficient matrix [4,5,12] like the Schur complement preconditioners [4] and augmentation preconditioners [5]. The role of preconditioners is to reduce the number of iterations required for convergence, while not increasing significantly the amount of computation required at each iteration. In addition, more iterative methods are proposed when $\mathcal{A}$ is symmetric, such as the Uzawa method, the parameterized inexact Uzawa methods [6–8], the GSOR method [9], the HSS based methods [10–12], and some combined methods like the UZAWA–SOR method [13] or the UZAWA–HSS method

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[14], etc. These methods are stationary methods, which require much less computer memory than the Krylov subspace methods in actual implementation. However, they may be less efficient in some situations. Thus, when the matrix blocks A, B and C are large and sparse, iterative methods become more attractive than direct methods for solving the saddle point problem (1.1). But direct methods play an important role in the form of preconditioners embedded in an iterative framework.

In this paper, we construct a new preconditioned iterative method to solve more general situations for the indefinite saddle point problems in (1.1). The preconditioning matrix is used at first. Then we get the iteration matrix by splitting the preconditioned coefficient matrix and also give the corresponding convergence analysis.

The remainder of the paper is organized as follows. In Section 2, we propose a new preconditioned parameterized inexact Uzawa method for indefinite saddle point problems and analyze its convergence. In Section 3, we generalize the method for which A is nonsingular. We modify the preconditioning matrix and discuss its convergence. For guaranteeing convergence, we derive several algorithms by different choices of the parameters and matrices in Section 4. In Section 5, we use a numerical example to show the fast convergence of the method we proposed, which also shows that these new methods are efficient and powerful.

2. Preconditioned parameterized inexact Uzawa methods when A is symmetric positive definite

Firstly, for preconditioning the linear system (1.1) when A is symmetric positive definite, we give a nonsingular matrix P:

$$P = \begin{bmatrix} MA^{-1} & 0 \\ BA^{-1}C^TCA^{-1} & BA^{-1}C^T \end{bmatrix},$$

where $M \in R^{n\times n}$ is symmetric positive definite. Then we get a new linear system of the form

$$Ku = d, \tag{2.1}$$

where

$$K = PA = \begin{bmatrix} M & MA^{-1}B^T \\ 0 & BA^{-1}C^TCA^{-1}B^T \end{bmatrix},$$

and

$$d = Pb = \begin{bmatrix} MA^{-1}f \\ BA^{-1}C^TCA^{-1}f + BA^{-1}C^Tg \end{bmatrix}.$$

After preconditioning, we found that K is nonsingular and consider the following matrix splitting:

$$K = \begin{bmatrix} M+Q_1 & 0 \\ Q_3 & Q_2 \end{bmatrix} - \begin{bmatrix} Q_1 & -MA^{-1}B^T \\ Q_3 & Q_2 - BA^{-1}C^TCA^{-1}B^T \end{bmatrix},$$

where $M+Q_1$ and Q_2 are nonsingular, and $Q_3 \in R^{m\times n}$ is arbitrary.

According to the preceding splitting, the generalized iterative method can be defined as

$$\begin{bmatrix} M+Q_1 & 0 \\ Q_3 & Q_2 \end{bmatrix}\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} Q_1 & -MA^{-1}B^T \\ Q_3 & Q_2 - BA^{-1}C^TCA^{-1}B^T \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix} + d. \tag{2.2}$$

To deal with $Q_2 - BA^{-1}C^TCA^{-1}B^T$, we assume $BA^{-1}C^TCA^{-1}B^T = \delta Q_2$ with $\delta > 0$, where Q_2 is symmetric positive definite. Then (2.2) can be redefined as

$$\begin{bmatrix} M+Q_1 & 0 \\ Q_3 & Q_2 \end{bmatrix}\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} Q_1 & -MA^{-1}B^T \\ Q_3 & (1-\delta)Q_2 \end{bmatrix}\begin{bmatrix} x_n \\ y_n \end{bmatrix} + d, \tag{2.3}$$

or equivalently, the system can be written as

$$\begin{cases} x_{n+1} = x_n + (M+Q_1)^{-1}MA^{-1}(f - Ax_n - B^Ty_n), \\ y_{n+1} = (1-\delta)y_n + Q_2^{-1}[Q_3(-x_{n+1}+x_n) + BA^{-1}C^T(CA^{-1}f+g)]. \end{cases} \tag{2.4}$$

Note the corresponding iteration matrix of the iterative scheme (2.3) is given by

$$W = \begin{bmatrix} M+Q_1 & 0 \\ Q_3 & Q_2 \end{bmatrix}^{-1}\begin{bmatrix} Q_1 & -MA^{-1}B^T \\ Q_3 & (1-\delta)Q_2 \end{bmatrix}. \tag{2.5}$$

If $\rho(W)$ denotes the spectral radius of the iterative matrix W, then the iterative scheme (2.4) converges if and only if $\rho(W) < 1$. Hence, let λ be an eigenvalue of W and $[x^H, y^H]^H$ be its corresponding eigenvector, where $x \in C^n$ and $y \in C^m$. Then we get

$$W\begin{bmatrix} x \\ y \end{bmatrix} = \lambda\begin{bmatrix} x \\ y \end{bmatrix},$$