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An inequality between the edge-Wiener index and the Wiener index of a graph

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ABSTRACT

The Wiener index $W(G)$ of a connected graph G is defined to be the sum $\Sigma_{u,v} d(u,v)$ of distances between all unordered pairs of vertices in G. Similarly, the edge-Wiener index $W_e(G)$ of G is defined to be the sum $\Sigma_{e,f} d(e,f)$ of distances between all unordered pairs of edges in G, or equivalently, the Wiener index of the line graph $L(G)$. Wu (2010) showed that $W_e(G) \geq W(G)$ for graphs of minimum degree 2, where equality holds only when G is a cycle. Similarly, in Knor et al. (2014), it was shown that $W_e(G) \geq \frac{\delta^2-1}{4} W(G)$ where δ denotes the minimum degree in G. In this paper, we extend/improve these two results by showing that $W_e(G) \geq \frac{\delta^2}{4} W(G)$ with equality satisfied only if G is a path on 3 vertices or a cycle. Besides this, we also consider the upper bound for $W_e(G)$ as well as the ratio $\frac{W_e(G)}{W(G)}$. We show that among graphs G on n vertices $\frac{W_e(G)}{W(G)}$ attains its minimum for the star.

1. Introduction

For a graph G, let $\deg(u)$ and $d(u, v)$ denote the degree of a vertex $u \in V(G)$ and the distance between vertices $u, v \in V(G)$, respectively. Let $L(G)$ denote the line graph of G, that is, the graph with vertex set $E(G)$ and two distinct edges $e, f \in E(G)$ adjacent in $L(G)$ whenever they share an end-vertex in G. Furthermore, for $e, f \in E(G)$, we let $d(e, f)$ denote the distance between e and f in the line graph $L(G)$.

In this paper we consider three important graph invariants, called *Wiener index* (denoted by $W(G)$ and introduced in [36]), *edge-Wiener index* (denoted by $W_e(G)$ and introduced in [21]) and *Gutman index* (denoted by $\mathrm{Gut}(G)$ and introduced in [12]), which are defined as follows:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v),$$

$$W_e(G) = \sum_{\{e,f\} \subseteq E(G)} d(e,f),$$

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$$\mathrm{Gut}(G) = \sum_{\{u,v\}\subseteq V(G)} \deg(u)\deg(v)\, d(u,v).$$

Observe that the edge-Wiener index of G is nothing but the Wiener index of the line graph $L(G)$ of G. Note also that in the literature a slightly different definition of the edge-Wiener index is sometimes used; for example, in [20] edge-Wiener index is defined to be $W_e(G) + \binom{n}{2}$ where $W_e(G)$ is defined as above and n is the order of G.

The Wiener index and related distance-based graph invariants have found extensive application in chemistry, see for example [14,15,34], and [2,8,16–18,30,31]] for some recent studies. The Wiener index of a graph was investigated also from a purely graph-theoretical point of view (for early results, see for example [9,33], and [4,25,26,38] for some surveys). Generalizations of Wiener index and relationships between these were studied in a number of papers (see for example [3,5,6,20]), and relationships between generalized graph entropies and the Wiener index (among other related topological indices) were established in [28]. New results on the Wiener index are constantly being reported, see for instance [10,19,23,29,35] for recent research trends.

Wu [37] showed that $W_e(G) \geq W(G)$ for graphs of minimum degree 2 where equality holds only when G is a cycle. Similarly, in [24] it was shown that $W_e(G) \geq \frac{\delta^2-1}{4} W(G)$ where δ denotes the minimum degree in G. In this paper, we improve these two results by showing that $W_e(G) \geq \frac{\delta^2}{4} W(G)$ with equality satisfied only if G is a path on 3 vertices or a cycle. One of the closely related distance-based graph invariant is the Szeged index [11], and a relation between the Szeged index and its edge version was recently established in [27].

In [3] it was proved that $W_e(G) \leq \frac{2^2}{5^5} + O(n^{9/2})$ for graphs of order n. Using the result of [32] we improve this bound to $W_e(G) \leq \frac{2^2}{5^5} + O(n^4)$. We also consider the ratio $\frac{W_e(G)}{W(G)}$ and show that this ratio is minimum if G is the star S_n on n vertices. Consequently, if G is a graph on n vertices, then $\frac{W_e(G)}{W(G)} \geq \frac{n-2}{2(n-1)}$.

2. Distances, average distance and D_α relations

Note that for any two distinct edges $e = u_1u_2$ and $f = v_1v_2$ in $E(G)$, the distance between e and f equals

$$d(e,f) = \min\{d(u_i, v_j) : i, j \in \{1,2\}\} + 1. \tag{1}$$

In the case when e and f coincide, we have $d(e, f) = 0$. In addition to the distance between two edges we will also consider the *average distance* between the endpoints of two edges, defined by

$$s(u_1u_2, v_1v_2) = \frac{1}{4}\Big(d(u_1,v_1) + d(u_1,v_2) + d(u_2,v_1) + d(u_2,v_2)\Big).$$

Notice that $s(e, f) = \frac{1}{2}$ when e and f coincide. The average distance of endpoints is in an interesting relationship with the Gutman index of a graph. Namely, if one likes to consider the version of edge-Wiener index where the distances between edges are replaced by the average distances of their endpoints, then what one gets is essentially the Gutman index, see Lemma 1.

A variation to the following result was mentioned in [24,37], where the sum in (2) is taken over all ordered pairs of edges. In our case the sum runs over all 2-element subsets of $E(G)$.

Lemma 1. *Let G be a connected graph. Then*

$$\sum_{\{e,f\}\subseteq E(G)} s(e,f) = \frac{1}{4}\Big(\mathrm{Gut}(G) - |E(G)|\Big). \tag{2}$$

Proof. Consider the sum on the left-hand side of (2). We can rewrite it as

$$\frac{1}{4} \sum_{\{uw,vz\}\subseteq E(G)} \Big(d(u,v) + d(u,z) + d(w,v) + d(w,z)\Big).$$

Now, for any two non-adjacent vertices of G, say u and v, the distance $d(u, v)$ appears in the above sum precisely once for each pair of edges, where one of these edges is incident with u and the other is incident with v. Thus, $d(u, v)$ appears in total precisely $\deg(u)\cdot\deg(v)$ times. And, if u and v are two adjacent vertices of G, then the distance $d(u, v) = 1$ appears in that sum precisely $\deg(u)\cdot\deg(v) - 1$ times. Thus, the above sum equals

$$\frac{1}{4}\Bigg[\sum_{uv\notin E(G)} \deg(u)\deg(v)d(u,v) + \sum_{uv\in E(G)} \Big(\deg(u)\deg(v) - 1\Big)d(u,v)\Bigg],$$

which is the right-hand side of (2). □

Now we define the following notions. Let G be a graph. For a pair of edges e and f of G we define the *difference*

$$D(e,f) = d(e,f) - s(e,f).$$

Moreover, if $D(e, f) = \alpha$, we say that e, f form a pair of type D_α or that the pair e, f *belongs to the set* D_α. Note that if $e = f$, then $D(e, f) = -\frac{1}{2}$. Denote by $\mathcal{I}$ the set $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. Note that $\sum_{\alpha\in\mathcal{I}} |D_\alpha| = \binom{|E(G)|}{2}$. Next easy lemma shows that $D(e, f) \in \mathcal{I}$ whenever $e \neq f$.