Applied Mathematics and Computation 269 (2015) 775–791

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

The high-order compact numerical algorithms for the two-dimensional fractional sub-diffusion equation☆

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ARTICLE INFO

Keywords:
Two-dimensional fractional sub-diffusion equation
Numerical schemes
High-order
Discrete energy method
Stability
Convergence

ABSTRACT

In this paper, performing the average operators on the space variables, a numerical scheme with third-order temporal convergence for the two-dimensional fractional sub-diffusion equation is considered, for which the unconditional stability and convergence in $L_1(L_\infty)$-norm are strictly analyzed for $\alpha \in (0, 0.9569347]$ by using the discrete energy method. Therewith, adding small perturbation terms, we construct a compact alternating direction implicit difference scheme for the two-dimensional case. Finally, some numerical results have been given to show the computational efficiency and numerical accuracy of both schemes for all $\alpha \in (0, 1)$.

1. Introduction

Fractional-order derivatives and integrals can depict heavy-tailed motions more accurately with their memory and hereditary properties of different substances, while traditional partial differential equations may not be adequate to describe those phenomena. Hence, fractional models are highly considered in various fields of science and engineering in recent few decades. Though, there exist some methods for solving analytical solutions of fractional partial differential equations (FPDEs), like the Fourier transform method, the Green function method, the Laplace transform method, variable separation method and so on, they are hardly to be adopted to compute in practical models since their forms of infinite series, and the exact solutions are unavailable for most FPDEs. So the effective numerical methods to solve FPDEs are keenly focused on, such as the finite element method [1–3], the spectral method [4–6], meshless method [7,8], etc. The finite difference scheme, known as a popular numerical algorithm, has been extensively investigated in many fractional models.

Anomalous diffusion phenomena have been observed in transport processes through complex and/or disordered systems in highly non-homogeneous media, which are qualitatively different from the standard Brownian motion. The diffusion equations containing fractional derivatives in time and/or in space are usually adopted to model phenomena of anomalous transport in physics and describe fractional random walk, wave propagation, fractal, etc. For the time-fractional sub-diffusion equations, the relatively earlier idea on the numerical schemes may be attributed to the two approximations to fractional time derivatives: the $L1$ formula with $2-\alpha$ order and the shifted Grünwald–Letnikov formula with first order. The interested readers can turn to publications [9–20].

It should be pointed out that the main challenges of solving the time-fractional diffusion equations numerically are time-consuming and high-storage capacity, since the numerical solutions at all previous time levels need to be preserved and involved

☆ The research is supported by National Natural Science Foundation of China, Grant no. 11271068.
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http://dx.doi.org/10.1016/j.amc.2015.07.088
0096-3003/

in the calculation when computing the solutions at a certain time level. Hence, the construction of high-accuracy numerical schemes is an important task. Recently, the methods on improving numerical accuracy of approximations cover two ways: increasing the spatial convergence order and increasing the temporal convergence order. The direct applications of the compact difference scheme [21–23] and spectral method [5,6] for the time-fractional diffusion equations can improve the spatial accuracy. However, it is more difficult to enhance the numerical accuracy in time due to the kernel singularity of fractional time derivatives. Zhuang et al. [24] integrated the anomalous sub-diffusion equation and constructed an implicit numerical scheme with global convergence order $O(\tau + h^2)$. Therewith, two solution techniques, the extrapolation method and linear interpolation for the fractional integral operator of the integrated equation, were presented for improving temporal convergence accuracy to second-order, which is demonstrated by numerical examples. Gao et al. [25] proposed a $L1-2$ formula based on the typical $L1$ approximation to discretize the Caputo fractional derivative with $3-\alpha$ order for $0 < \alpha < 1$, then the formula was applied to construct a class of difference scheme for the one-dimensional Caputo-type fractional sub-diffusion equation, where there is no theoretical analysis on convergence and stability of the scheme yet. Zhao et al. [26] presented second-order approximations for variable order fractional derivatives and gave some applications. Alikhanov [27] came up with a new difference analog of the Caputo fractional derivative and on its basis some difference schemes generating approximations of the second- and fourth-order in space and the second-order in time for the one-dimensional time-fractional diffusion equation with variable coefficients were considered. The stability and convergence in L_2-norm was proved. By weighting the shifted Grünwald–Letnikov formula and choosing shifts $(p, q) = (0, -1)$, a numerical scheme with second-order convergence in time was established for solving the modified Riemann–Liouville-type anomalous fractional sub-diffusion equation [28]. Based on Lubich's operator, Li and Ding [29] derived a second-order difference approximation for the Riemann–Liouville fractional derivative, but they just proved the unconditional stability and convergence in L_2-norm for $\alpha \in [\frac{3}{8}, 1)$. Recently, Zeng et al. [3] have proposed two improved algorithms for the time-fractional subdiffusion equation with time discretization by the fractional linear multistep methods and space discretization by finite element method. The global convergence order is $O(\tau^2 + h^{r+1})$, where τ and h are the step sizes in time and space, respectively, and r is the degree of the piecewise polynomial space.

To the author's best knowledge on the finite difference method for the two-dimensional Caputo-type time-fractional subdiffusion equation, the numerical schemes involving high convergence order in time are relatively scarce. Therefore, developing a high-accuracy numerical algorithm both in time and in space for the two-dimensional problem is a piece of valuable work, due to the history dependence and nonlocal character of the time fractional derivatives. The key point of this article is based on the recent work by Ji and Sun [30], where a third-order approximation to the time fractional derivative was derived. The extension to the two-dimensional case is taken into account here. The unconditional stability and convergence in the $L_1(L_\infty)$-norm are analyzed by discrete energy method, where the concrete verification skills are fundamentally different from [30] because of the disparity of the L_∞-norm embedded inequality between one-dimensional case and two-dimensional case. According to the present theoretical results, the quadratic form, generated from the coefficients of the third-order approximation [30] to the Caputo fractional derivative, is semi-positive definite for $\alpha \in (0, 0.9569347]$. However, the numerical results demonstrate that the difference scheme for the two-dimensional fractional sub-diffusion equation is third-order convergence in time for all $\alpha \in (0, 1)$. Meanwhile, a compact alternating direction implicit difference scheme is presented by adding small perturbation terms, which is tested by the numerical example.

The outline of this paper is arranged as follows. In Section 2, some definitions and lemmas are presented, which are used in the subsequent discussion. In Section 3, a compact difference scheme for the two-dimensional fractional sub-diffusion equation is proposed. The unconditional stability and convergence are rigorously analyzed in Section 4 by discrete energy method. In Section 5, adding small perturbation terms on the presented scheme in Section 3, a compact alternating direction implicit difference scheme is derived. The numerical results demonstrate the validity of the theoretical results and test the convergence orders of both schemes in Section 6. Some conclusions are drawn in the end.

2. Definitions and lemmas

We begin with some definitions of fractional derivatives and fractional integral of order $\alpha \in \mathbb{R}_+$. The interested readers can refer to the textbooks [31–33].

Definition 2.1. Suppose $f(t) \in L_1[a, b]$(a is a finite real number or $-\infty$). For $a \le t \le b$, the operator ${}_a\mathcal{D}_t^\alpha f(t)$ defined by

$$ {}_a\mathcal{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\xi)}{(t-\xi)^{\alpha+1-n}} d\xi, \quad n-1 < \alpha < n, $$

is called the Riemann–Liouville fractional derivative of order α, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. Suppose $f(t) \in C^1[0, T]$. For $0 \le t \le T$, the operator ${}_0^C\mathcal{D}_t^\alpha f(t)$ defined by

$$ {}_0^C\mathcal{D}_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\xi)}{(t-\xi)^\alpha} d\xi, \quad 0 < \alpha < 1, $$

is called the Caputo fractional derivative of order α.