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Boundary value problems with higher order Lipschitz boundary data for polymonogenic functions in fractal domains

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ABSTRACT

In this note we consider certain jump problem for poly-monogenic functions in fractal domains with higher order Lipschitz boundary data. This is accomplished by using a higher order Teodorescu operator which replaces the expected surface integral. Also, we give out the uniqueness of solutions basing the work on the method of removable singularities for monogenic functions making use of a Dolzhenko type theorem.

1. Introduction

Considerable work has been done in the study of boundary value problems for elliptic differential equations of second order in the complex plane, see for instance [3,23–25].

The last few decades have witnessed a surge of wide discussion on boundary value problems for different classes of higher order complex partial differential equations of arbitrary order, examples are poly- and meta-analytic, poly-harmonic functions, etc. (see, for instance, references [2,7,12,21] or elsewhere). They have abundant applications in mathematical physics and engineering.

As an elegant generalization of analytic functions from the complex plane to higher-dimensions, Clifford analysis [6] concentrates on the study of the so-called monogenic functions, i.e. null solutions to the Dirac operator $\mathcal{D}$. Moreover, due to the fact that $\mathcal{D}$ factorizes the Laplacian, Clifford analysis can be seen as a refinement of classic harmonic analysis.

In the recent works [5,8,14,16–18,27] with the aid of Clifford analysis techniques different kinds of boundary value problems (basically consistent in the study of some jump conditions, e.g., Riemann–Hilbert type problems) for the iterated Dirac equations $\mathcal{D}^k f = 0$ were investigated. However, it is worth pointing out that the study of those boundary value problems has been confined to domains with sufficiently smooth boundary. Of course, for the purpose of numerous applications, and for sake of pure mathematical generality, it would be of great interest to be able to lift these geometric restrictions.

The aim of this paper is to enlarge the class of domains, allowing the inclusion of those with fractal boundaries, where it is possible to study boundary value problems for poly-monogenic functions essentially reducible to an examination of jump conditions.

The results contained in this work can be applied in practical electromagnetic wave propagation and scattering phenomena in fractal media as well as fractal antennas design, see [19] and references there in. The first ideas related to these devices were proposed more than 15 years ago, however the electromagnetic theory of these antennas is not well-developed so far.

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2. Preliminaries

2.1. Lipschitz classes and Whitney extension theorem

Let $\mathbf{E}$ be a closed subset of $\mathbb{R}^m$. The Whitney data (or smooth functions in the sense of Whitney) is, by definition, a collection of continuous functions $\{f^{(j)} : \mathbf{E} \mapsto \mathbb{R},\ |j| \le k\}$, $j := (j_1, j_2, \dots, j_m)$ with $|j| := j_1 + j_2 + \cdots + j_m \le k$, which satisfy the compatibility conditions that for each multi-index j, the differences

$$R_j(x,y) := f^{(j)}(x) - \sum_{|l|\le k-|j|} \frac{f^{(j+l)}(y)}{l!}(x-y)^l, \quad x, y \in \mathbf{E}$$

should be sufficiently small as specified by

$$|R_j(x,y)| = o(|x-y|^{k-|j|}), \quad x,y \in \mathbf{E}, \quad |x-y| \to 0$$

In 1934 Hassler Whitney, one of the most influential mathematicians of the twentieth century [15], proved in [26] that such a collection can be extended as a C^k-smooth function on $\mathbb{R}^m$. This means that there exists a smooth function $f : \mathbb{R}^m \mapsto \mathbb{R}$ such that for any multi-index j the restriction on $\mathbf{E}$ of $\partial^{(j)} f$ coincides with $f^{(j)}$. In particular $\partial^{(0)} f := f$ coincides with $f^{(0)}$ on $\mathbf{E}$.

For clarity, however, we record the multi-index notation

$$\partial^{(j)} := \frac{\partial^{|j|}}{\partial x_1^{j_1} \partial x_2^{j_2} \dots \partial x_m^{j_m}}.$$

Notice that we cannot say initially that the $f^{(j)}$ functions are the derivatives of $f^{(0)}$ on $\mathbf{E}$ due the closeness of $\mathbf{E}$, and so differentiation might not makes sense. However, it is possible to slightly enhance the above result further by allowing an arbitrary not integer exponent $k + \alpha$, which gives rise to the higher order Lipschitz classes, see [22].

Let k be a non-negative integer and $0 < \alpha \le 1$. We shall say that a function f, defined in $\mathbf{E}$, belongs to $\mathrm{Lip}(\mathbf{E}, k+\alpha)$ if there exist bounded functions $f^{(j)}$, $0 < |j| \le k$, defined on $\mathbf{E}$, with $f^{(0)} = f$, and so that

$$R_j(x,y) = f^{(j)}(x) - \sum_{|l|\le k-|j|} \frac{f^{(j+l)}(y)}{l!}(x-y)^l, \quad x, y \in \mathbf{E} \tag{1}$$

satisfies

$$|R_j(x,y)| = O(|x-y|^{k+\alpha-|j|}), \quad x,y \in \mathbf{E}, \quad |j| \le k. \tag{2}$$

It should be remarked that the function $f^{(0)} = f$ does not necessarily determine the functions $f^{(j)}$. For that reason an element of $\mathrm{Lip}(\mathbf{E}, k+\alpha)$ has to be interpreted as a collection $\{f^{(j)} : \mathbf{E} \mapsto \mathbb{R},\ |j| \le k\}$.

When $k = 0$, the Lipschitz class becomes the usual class $C_b^{0,\alpha}(\mathbf{E})$ of bounded Hölder continuous functions in $\mathbf{E}$.

Remark 2.1. We remark that for $\mathbf{E} = \mathbb{R}^m$ the functions $f^{(j)}$ are uniquely determined by $f^{(0)}$ and $\mathrm{Lip}(\mathbb{R}^m, k+\alpha)$ actually consists of continuous and bounded functions f with continuous and bounded partial derivatives $\partial^{(j)} f$ up to the order k. Moreover the functions $\partial^{(j)} f$, for $|j| = k$ belong to the space $\mathrm{Lip}(\mathbb{R}^m, \alpha)$.

Thus the Whitney extension theorem can be stated as follows.

Theorem 2.1 ([22]). *Let $f \in \mathrm{Lip}(\mathbf{E}, k+\alpha)$. Then, there exists a function $\tilde{f} \in \mathrm{Lip}(\mathbb{R}^m, k+\alpha)$ satisfying*

(i) $\tilde{f}|_{\mathbf{E}} = f^{(0)}$, $\partial^{(j)} \tilde{f}|_{\mathbf{E}} = f^{(j)}$,
(ii) $\tilde{f} \in C^\infty(\mathbb{R}^m \setminus \mathbf{E})$,
(iii) $|\partial^{(j)} \tilde{f}(x)| \leqslant c\, dist(x, \mathbf{E})^{\alpha-1}$, *for* $|j| = k+1$ *and* $x \in \mathbb{R}^m \setminus \mathbf{E}$.

In fact, this extension result is based upon the Whitney decomposition of $\mathbb{R}^m \setminus \mathbf{E}$ by a disjoint union of countably many dyadic cubes, customarily denoted by the letter Q. We shall use the symbol $|Q|$ to denote the diameter of Q.

We shall return to this central theorem in Section 3.

3. Polymonogenic functions

Denote by $e_1, e_2, \dots, e_n$ an orthonormal basis of $\mathbb{R}^n$ that satisfy

$$e_i^2 = -1,\ e_i e_{i'} = -e_{i'} e_i, \quad i, i' = 1, 2, \dots, n,\ i < i'.$$

Thus the Euclidean space

$$\mathbb{R}^n = \{x = x_1 e_1 + x_2 e_2 + \cdots x_n e_n,\ x_i \in \mathbb{R}, i = 1, 2, \dots, n\}$$

is embedded in the real Clifford algebra $\mathbb{R}_{0,n}$ generated by $e_1, e_2, \dots, e_n$ over the field of real numbers $\mathbb{R}$.

Similarly, $\mathbb{R}^{n+1}$ is embedded in $\mathbb{R}_{0,n}$ by identifying $x \in \mathbb{R}^{n+1}$ with the so-called paravector $x = x_0 + x_1 e_1 + x_2 e_2 + \cdots x_n e_n$.