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Oscillation criteria for second order nonlinear delay dynamic equations on time scales☆

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ABSTRACT

By employing the generalized Riccati transformation, we establish some new oscillation criteria for second order delay dynamic equations on time scales. The obtained results essentially improve the well-know oscillation results for half-linear dynamic equations such as Kamenev-type and Philos-type oscillation criteria. Some interesting examples are given to illustrate the versatility of our results.

1. Introduction

The theory of time scales has received a lot of attention since Hilger introduced the theory of time scale which was expected to unify continuous and discrete calculus. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. In recent years, there has been an increasing interest in studying oscillation and nonoscillation of solutions of dynamic equations on time scales. We refer the readers to the monographs [5,21], the papers [1–3,6,7,9–15,17–20,22–24] and the references cited therein.

In this paper, we investigate oscillation of second order nonlinear delay dynamic equations of the following form

$$[r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)]^{\Delta}+p(t)f(x(\tau(t)))=0, \quad \gamma>0. \tag{1.1}$$

Throughout this paper, we shall assume that the time scale T of the form $[t_{-1},\infty)_{\mathbb{T}}$, where $t_{-1}=\inf_{t\geq t_0}\tau(t)$, and is unbounded above. Furthermore, we assume that

(A_1) $p\in C_{rd}(\mathbb{T},(0,\infty))$, $r\in C_{rd}(\mathbb{T},(0,\infty))$ satisfies

$$\int_{t_0}^{\infty}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}\Delta s=\infty;$$

(A_2) the delay function $\tau\in C_{rd}(\mathbb{T},\mathbb{T})$ satisfies $\tau(t)\leq t$ and $\lim\limits_{t\to\infty}\tau(t)=\infty$;

(A_3) $f\in C(\mathbb{R},\mathbb{R})$ satisfies $f(y)/(|y|^{\gamma-1}y)\geq K>0$ for $y\neq 0$, where K is a constant.

If $f(x(\tau(t)))=|x(\tau(t))|^{\gamma-1}x(\tau(t))$, then (1.1) is called half-linear dynamic equation or p-Laplace equation. Since half-linear equations are able to model all kinds of real world problems such as continuum mechanics, non-Newtonian fluid theory, and

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the turbulent flow of a polytrophic gas in a porous medium, the study of half-linear equations has become an important area of research. For more details, see the monograph [4].

Note that the condition $f(y)/(|y|^{\gamma-1}y) \geq K > 0$ for $y \neq 0$ implies $f(y)y > 0$ for $y \neq 0$ which is necessary in [1,13]. A solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Erbe et al. [13] and Agarwal et al. [1] studied (1.1) for the special case $\gamma = 1$:

$$\left[r(t)x^{\Delta}(t)\right]^{\Delta} + p(t)f(x(\tau(t))) = 0, \tag{1.2}$$

where $f(y)/y \geq K > 0$. In order to establish new oscillation results for (1.2), they used the generalized Riccati transformation and introduced the class of functions $\Re_1$ as follows: $H_1 \in \Re_1$ if H_1 is defined for $t_0 \leq s \leq \sigma(t), t, s \in [t_0, \infty)_{\mathbb{T}}, H_1(t, s) \geq 0, H_1(\sigma(t), t) = 0, H_1^{\Delta_s}(t, s) \leq 0$ for $s \geq t_0$, and for each fixed t, $H_1^{\Delta_s}(t, s)$ is delta integrable with respect to s.

Saker [19], Hassan [18] and Agarwal et al. [1,6] considered the half-linear dynamic equation

$$\left[r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)\right]^{\Delta} + p(t)|x(t)|^{\gamma-1}x(t) = 0. \tag{1.3}$$

To present oscillation criteria for (1.3), they utilized the Riccati substitution and defined the functions space $\Re_2$ as follows: $H_2 \in \Re_2$ if $H_2 : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ and satisfies $H_2(t, t) = 0, t \geq t_0, H_2(t, s) > 0, t > s \geq t_0$.

Motivated by [1,6,13,18,19] we employ the generalized Riccati transformation technique to establish new oscillation criteria for (1.1), which improve and generalize the results in Saker[19], Erbe et al. [13], Hassan [18] and Agarwal et al. [1,6] for $\int_{t_0}^{\infty} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s = \infty$.

The paper is organized as follows: In Section 2, we present some preliminaries and important estimates, and then give several oscillatory results. In Section 3, we illustrate the main results by three examples.

2. Main results

In order to prove our main results, we establish some fundamental results in this section. Now, for $a \in [t_0, \infty)_{\mathbb{T}}$, we introduce the auxiliary functions

$$R(t, a) = \int_a^t \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \Delta s, \quad \eta_1(t, a) = \frac{R(\tau(t), a)}{R(\sigma(t), a)}, \quad \eta_2(t, a) = \frac{R(t, a)}{R(\sigma(t), a)}. \tag{2.1}$$

Before stating our main results, we begin with the following lemma which will play an important role in the proofs of our main results.

Lemma 2.1. *Assume that conditions $(A_1) - (A_3)$ hold. Let $x(t)$ be an eventually positive solution of (1.1). Then, there exists some $T > t_0$ large enough such that for all $t \geq T$, $[r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)]^{\Delta} < 0, x^{\Delta}(t) > 0, x(t) \geq [r(t)]^{\frac{1}{\gamma}}R(t, T)x^{\Delta}(t)$, and*

$$x(\tau(t)) \geq \eta_1(t, T)x^{\sigma}(t), \quad x(t) \geq \eta_2(t, T)x^{\sigma}(t).$$

Proof. Suppose that $x(t)$ is an eventually positive solution of (1.1), i.e. there exists a $T \in \mathbb{T}$ large enough such that $x(t) > 0$ for all $t \geq T$. Similar to [18, Lemma 2.1], it is not difficult to derive $[r(t)|x^{\Delta}(t)|^{\gamma-1}x^{\Delta}(t)]^{\Delta} < 0, x^{\Delta}(t) > 0$, and $x(t) \geq [r(t)]^{\frac{1}{\gamma}}R(t, T)x^{\Delta}(t)$. Thus, we have $(x(t)/R(t, T))^{\Delta} \leq 0$, which implies that

$$\frac{x(\tau(t))}{R(\tau(t), T)} \geq \frac{x(t)}{R(t, T)} \geq \frac{x(\sigma(t))}{R(\sigma(t), T)}.$$

The proof is complete. □

In the sequel, we use the following notations:

$$\beta_1(t) = \begin{cases} \eta_2(t, T), & 0 < \gamma < 1, \\ \eta_2^{\gamma}(t, T), & \gamma \geq 1, \end{cases}$$

$$d_+(t) := \max\{0, d(t)\}, \quad \varphi(t) = \delta^{\sigma}(t)[Kp(t)\eta_1^{\gamma}(t, T) - (r(t)a(t))^{\Delta} + r(t)\beta_1(t)a^{1+\frac{1}{\gamma}}(t)],$$

$$\varphi_1(t) = \frac{\delta^{\Delta}(t)}{\delta(t)} + (\gamma + 1)a^{\frac{1}{\gamma}}(t)\frac{\beta_1(t)\delta^{\sigma}(t)}{\delta(t)}, \quad \varphi_2(t) = \frac{\gamma\beta_1(t)\delta^{\sigma}(t)}{r^{\frac{1}{\gamma}}(t)\delta^{1+\frac{1}{\gamma}}(t)},$$

where $\eta_1(t, T), \eta_2(t, T)$ is given in (2.1), $\delta(t)$ and $a(t) > -1/[r(t)R^{\gamma}(t, T)]$ will be specified later.

Using the Pötzsche chain rule [8, Theorem 1.90], it is easy to obtain the following lemma.

Lemma 2.2. *Let $\gamma > 0$ be a constant. Suppose that $g : \mathbb{T} \to \mathbb{R}$ is positive and delta differentiable, and there exists some $t_* \in \mathbb{T}$ such that $g^{\Delta}(t) \geq 0$ for all $t \geq t_*$. Then, for all $t \geq t_*$,*

$$((g(t))^{\gamma})^{\Delta} \geq \begin{cases} \gamma(g^{\sigma}(t))^{\gamma-1}g^{\Delta}(t), & 0 < \gamma < 1; \\ \gamma(g(t))^{\gamma-1}g^{\Delta}(t), & \gamma \geq 1. \end{cases}$$