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# Graphical representations for the homogeneous bivariate Newton's method

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A B S T R A C T

In this paper we propose a new and effective strategy to apply Newton's method to the problem of finding the intersections of two real algebraic curves, that is, the roots of a pair of real bivariate polynomials. The use of adequate homogeneous coordinates and the extension of the domain where the iteration function is defined allow us to avoid some numerical difficulties, such as divisions by values close to zero. In fact, we consider an iteration map defined on a real augmented projective plane. So, we obtain a global description of the basins of attraction of the fixed points associated to the intersection of the curves. As an application of our techniques, we can plot the basins of attraction of the roots in the following geometric models: hemisphere, hemicube, Möbius band, square and disk. We can also give local graphical representations on any rectangle of the plane.



## Introduction

In this work we explore some topological and numerical aspects of the problem of finding the solutions of a system of non-linear equations $F(x, y) = 0$, where $F : \mathbb{R}^2 \to \mathbb{R}^2$. In particular, we consider the case $F(x, y) = (P(x, y), Q(x, y))$, where $P(x, y)$ and $Q(x, y)$ are two polynomial in two variables:

$$\begin{cases} P(x, y) = 0, \\ Q(x, y) = 0. \end{cases} \tag{1}$$

There is a lot of applications that require the solution of such systems, such as the intersection of two plane curves, the determination of the closest point of a curve to a given point and many other geometrical problems coming from different disciplines [15]. A first approach in the solution of this problem is the use of the classical Newton's method for systems. Under appropriate conditions, this method generates a sequence of approximations to a solution. The Newton sequence is defined as follows:

$$X_{n+1} = X_n - F'(X_n)^{-1}F(X_n), \quad X_0 = (x_0, y_0), \tag{2}$$

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where $(x_0, y_0)$ is the initial iterate and $F'(X)$ denotes the Jacobian matrix at a point $X$. In our case,

$$F'(X) = F'(x, y) = \begin{pmatrix} P_x(x, y) & P_y(x, y) \\ Q_x(x, y) & Q_y(x, y) \end{pmatrix}.$$

Alternatively, we can find each Newton step solving first the linear system

$$F'(X_n)(U_n) = -F(X_n) \tag{3}$$

and next defining $X_{n+1} = X_n + U_n$.

If we apply Newton's method (2) to the problem (1) we obtain two bivariate rational maps

$$f(x, y) = \frac{H_1(x, y)}{J_1(x, y)}, \quad g(x, y) = \frac{H_2(x, y)}{J_1(x, y)} \tag{4}$$

where the numerators $H_1(x, y)$, $H_2(x, y)$ and the common denominator $J_1(x, y)$ are bivariate polynomials, that can be obtained explicitly in terms of the polynomials $P(x, y)$, $Q(x, y)$ and their partial derivatives (see Section 1).

In the course of the iteration process of this pair $(f, g)$, there could be overflow problems. For instance, if the value of the denominator of $f(x, y)$ at a point $(x_n, y_n)$ is zero and the value of the numerator is non zero,

$$J_1(x_n, y_n) = 0, \quad H_1(x_n, y_n) \neq 0,$$

there is a infinite value of $f(x_n, y_n)$, which is not in the codomain of the function $f$. Furthermore, when the values of the numerator and the denominator are simultaneously zero

$$H_1(x_n, y_n) = 0, \quad J_1(x_n, y_n) = 0,$$

one also has an indeterminated value of the rational function $f$. Similar difficulties may appear for the second rational function $g(x, y)$.

One of the aims of this work is to find a solution to these indefinite or indetermination problems that could permits us to continue with new iterations of the pair of bivariate rational maps. In this paper a simple solution to these difficulties is obtained by considering an extension of the domain and of the iteration map $(f, g)$.

We extend the initial domain of $(f, g)$ to $\mathbf{p}^{2+}(\mathbb{R})$ the augmented real projective plane (see Section 2.2). Now, if the evaluation at a point of the denominator gives a zero value, then an element of the extension (an "infinity point") can be taken as the image of this point. The function $(f, g)$ also has an extension to this augmented projective plane that is defined by a homogeneous triple of 3-variate polynomials. Then the extended function can be easily iterated without problems until we decide to stop the iteration process under some established criterion.

Our work studies the dynamics of maps defined on a real projective plane and we focus on a graphical study of the basins of fixed points of the iteration map. The boundary of these basins has a fractal structure that can be analyzed by using either Mandelbrot–Julia techniques or dynamical methods of Riemann surfaces. Recently many advances have been developed in the study of the fractal feature of generalized Mandelbrot–Julia sets [25,29,30]. There are also important contributions and results on dynamics and geometry of Riemann surfaces and their moduli spaces (see [7,8]), which are also related to the dynamics on a real projective plane.

In Section 1 we show that, when Newton's method is applied to find the roots of two real bivariate polynomial equations, a homogeneous triple of polynomials is canonically induced. In Section 2 we describe all the mathematical tools which are necessary for the design of an algorithm that will allow us the iteration of a function defined by a homogeneous triple of polynomials on $\mathbf{p}^{2+}(\mathbb{R})$. We have also implemented this process in a new program, written in *Mathematica*, which allows us to visualize the attraction basins of a list of fixed points. A description of these algorithms is given in section 3, and some applications of these algorithms and their implementations are given in Section 4. This section contains a graphical study of the attraction basins for the intersection points of a pair of real algebraic curves (with low degrees) when the bivariate Newton's method is applied.

One interesting novelty brought by the developed program is that it permits to plot spherical fractals (with antipodal symmetry) with the basins associated to the real roots of two equations given by a pair of real bivariate polynomials. This follows from the fact that one has a canonical surjective map from $S^2$ to the space $\mathbf{p}^{2+}(\mathbb{R})$. Moreover, since the disjoint union of a Möbius band and an infinity point is bijective to the real projective plane and the Möbius band can be obtained by the identification of a pair of opposite sides in a square, one can also plot the global attraction basins in either on a Möbius band or a square. For instante, for the pair of algebraic curves

$$x^2 + 4y^2 - 4 = 0, \quad 16x^4 + y^4 - 16 = 0 \tag{5}$$

one has four real intersection points an theirs global attraction basins are represented in four different colors in the Fig. 1.

Many other iterative methods induce a rational function defined on the complex projective line $\mathbf{p}^1(\mathbb{C})$, which can be used to find the roots of a complex polynomial. Several graphical approaches to the study of attraction basins and Julia sets of these numerical methods can be found in the literature. See for instance [18,24,33].

## 1. Bivariate Newton method and real homogeneous triples of polynomials

In this section we describe how a real homogeneous triple of polynomials can be associated to any pair of real bivariate polynomials when the multidimensional Newton's method (2) is applied to the problem (1).