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# Isogeometric Analysis of the Navier–Stokes equations with Taylor–Hood B-spline elements

Babak S. Hosseini [a,*], Matthias Möller [b], Stefan Turek [a]

[a] TU Dortmund, Institute of Applied Mathematics (LS III), Vogelpothsweg 87, 44227 Dortmund, Germany
[b] Delft University of Technology, Delft Institute of Applied Mathematics, Mekelweg 4, 2628 CD Delft, The Netherlands

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ABSTRACT

This paper presents our numerical results of the application of Isogeometric Analysis (IGA) to the velocity–pressure formulation of the steady state as well as to the unsteady incompressible Navier–Stokes equations. For the approximation of the velocity and pressure fields, LBB compatible B-spline spaces are used which can be regarded as smooth generalizations of Taylor–Hood pairs of finite element spaces. The single-step $\theta$-scheme is used for the discretization in time. The lid-driven cavity flow, in addition to its regularized version and flow around cylinder, are considered in two dimensions as model problems in order to investigate the numerical properties of the scheme.



## 1. Introduction

The Isogeometric Analysis technique, developed by Hughes et al. [7], is a powerful numerical technique aiming to bridge the gap between the worlds of computer-aided engineering (CAE) and computer-aided design (CAD). It combines the benefits of Finite Element Analysis (FEA) with the ability of an exact representation of complex computational domains via an elegant mathematical description in the form of uni-, bi- or trivariate non-uniform rational B-splines. Non-Uniform Rational B-splines (NURBS) are the de facto industry standard when it comes to modeling complex geometries, while FEA is a numerical approximation technique that is widely used in computational mechanics.

NURBS and FEA utilize basis functions for the representation of geometry and approximation of field variables, respectively. In order to close the gap between the two technologies, Isogeometric Analysis adopts the B-spline, NURBS or T-spline (see [7]) geometry as the computational domain and utilizes its basis functions to construct both trial and test spaces in the discrete variational formulation of differential problems. The usage of these functions allows the construction of approximation spaces exhibiting higher regularity ($\mathcal{C}^{\geqslant 0}$) which – depending on the problem to be solved – may be beneficial compared to standard finite element spaces. For instance, Cottrell, Hughes and Reali showed in their study of refinement and continuity in isogeometric structural analysis [8] that increased smoothness leads to a significant increase in accuracy for the problems of structural vibrations over the classical $\mathcal{C}^0$ continuous p-method of FEA. Isogeometric Analysis has been successfully applied to high order partial differential equations (PDEs) from a wide range of fields of computational mechanics. In fact, primal variational formulations of high order PDEs such as Navier–Stokes–Korteweg (3rd order spatial derivatives) or Cahn–Hilliard (4th order spatial derivatives) require piecewise smooth and globally $\mathcal{C}^1$ continuous basis functions. Note that

* Corresponding author.
*E-mail addresses:* babak.hosseini@math.tu-dortmund.de (B.S. Hosseini), m.moller@tudelft.nl (M. Möller), stefan.turek@math.tu-dortmund.de (S. Turek).

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the number of finite elements possessing $\mathcal{C}^1$ continuity and being applicable to complex geometries is already very limited in two dimensions [13,14]. The Isogeometric Analysis technology features a unique combination of attributes, namely, superior accuracy on degree of freedom basis, robustness, two- and three-dimensional geometric flexibility, compact support, and the possibility for $\mathcal{C}^{\geqslant 0}$ continuity [7].

This article is all about the application and assessment of the Isogeometric Analysis approach to fluid flows with respect to well known benchmark problems. We present our numerical results for the lid-driven cavity flow problem (including its regularized version) using different B-spline approximation spaces, and compare them to reference results from literature. Moreover, in addition to comparisons with classical references, we will whenever feasible take into consideration the results of two recently published articles [9,21] on the application of Galerkin-based IGA to the cavity flow problem. The analysis presented in [21] is based on a scalar stream function formulation of the Navier–Stokes equations, while [9] uses divergence-conforming B-splines which may be interpreted as smooth generalizations of Raviart–Thomas elements. We extend this Galerkin IGA-based row of results for cavity flow with data obtained from the application of smooth generalizations of Taylor–Hood elements. Despite the fact that investigations of lid-driven cavity type model problems do not necessarily reflect the current spirit of time, they are nonetheless a natural first choice in computational fluid dynamics when it comes to assessing the properties of a novel numerical technique.

Subsequent to lid-driven cavity, we eventually proceed to present and assess approximated physical quantities such as the drag and lift coefficients obtained for the flow around cylinder benchmark, whereby a multi-patch discretization approach is adopted. For the scenarios addressed, Isogeometric Analysis is applied to the steady-state as well as to the transient incompressible Navier–Stokes equations. For the equations under consideration are of nonlinear nature, we decided to provide a rather detailed insight concerning their treatment. The efficient solution of the discretized system of equations using iterative solution techniques such as, for instance, multigrid is not addressed in this paper. Preliminary research results are underway and will be presented in a forthcoming publication. In this numerical study, all systems of equations have been solved with a direct solver.

The outline of this paper is as follows: Section 2 is devoted to the introduction of the univariate and the multivariate (tensor product) B-spline and NURBS (non-uniform rational B-spline) basis functions, their related spaces, and the NURBS geometrical map **F**. This presentation is quite brief and notationally oriented. A more complete introduction to NURBS and Isogeometric Analysis can be found in [18,7,3]. Section 3 formalizes Taylor–Hood like discrete approximation spaces being used in different peculiarities throughout this article. Section 4 is dedicated to the presentation of the governing equations and their variational forms. The numerical results are showcased in Section 5. In particular, in Sections 5.2 and 5.4, numerical results of Isogeometric Analysis of lid-driven cavity flow and flow around cylinder are presented and compared with reference results from literature. Section 6 is dedicated to a short summary in addition to drawn conclusions.

## 2. Preliminaries

In order to fix the notation and for the sake of completeness, this section presents a brief overview of B-spline/NURBS basis functions and their corresponding spaces utilized in Isogeometric Analysis.

Galerkin-based Isogeometric Analysis adopts spline (B-spline/NURBS, etc.) basis functions for analysis as well as for the description of the geometry (computational domain). Just like in FEA, a discrete approximation space – based on the span of the basis functions in charge – is constructed and eventually used in the framework of a Galerkin procedure for the numerical approximation of the solution of partial differential equations.[1]

Recalling reference ($\tilde{\Omega}$) and physical domains ($\Omega$) in FEA, using B-splines/NURBS, one additional domain – the parametric spline domain ($\hat{\Omega}$) – needs to be considered as well (see Fig. 1). We follow this requirement and present an insight in the traits of spline-based discrete approximation spaces in the sequel.

Given two positive integers $p$ and $n$, we introduce the ordered knot vector

$$\Xi := \{0 = \xi_1, \xi_2, \ldots, \xi_m = 1\}, \tag{1}$$

whereby repetitions of the $m = n + p + 1$ knots $\xi_i$ are allowed: $\xi_1 \leqslant \xi_2 \leqslant \ldots \leqslant \xi_m$. Note that in (1) the values of $\Xi$ are normalized to the range $[0, 1]$ merely for the sake of clarity and not restricted in range otherwise. Besides, we assume that $\Xi$ is an open knot vector, that is, the first and last knots have multiplicity $p + 1$:

$$\Xi = \{\underbrace{0, \ldots, 0}_{p+1}, \xi_{p+2}, \ldots, \xi_{m-p-1}, \underbrace{1, \ldots, 1}_{p+1}\}.$$

Let the (univariate) B-spline basis functions of degree $p$ (order $p + 1$) be denoted by $B_{i,p}(\xi)$, for $i = 1, \ldots, n$. Then, the $i$th B-spline basis function is a piecewise polynomial function and it is recursively defined by the Cox-de Boor recursion formula:

[1] We point out on a side note that IGA is not restricted to the Galerkin framework and has as a matter of fact been successfully used with Collocation techniques as well, see for instance [2,20].