Applied Mathematics and Computation 267 (2015) 294–303

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Improved accuracy for time-splitting methods for the numerical solution of parabolic equations

A. Arrarás *, L. Portero

Departamento de Ingeniería Matemática e Informática, Universidad Pública de Navarra, Edificio Las Encinas, Campus de Arrosadía, 31006 Pamplona, Spain

ARTICLE INFO

Keywords:
Alternating direction implicit
Domain decomposition
Partition of unity
Splitting error
Time-splitting method

ABSTRACT

In this work, we study time-splitting strategies for the numerical approximation of evolutionary reaction–diffusion problems. In particular, we formulate a family of domain decomposition splitting methods that overcomes some typical limitations of classical alternating direction implicit (ADI) schemes. The splitting error associated with such methods is observed to be $\mathcal{O}(\tau^2)$ in the time step τ. In order to decrease the size of this splitting error to $\mathcal{O}(\tau^3)$, we add a correction term to the right-hand side of the original formulation. This procedure is based on the improved initialization technique proposed by Douglas and Kim in the framework of ADI methods. The resulting non-iterative schemes reduce the global system to a collection of uncoupled subdomain problems that can be solved in parallel. Computational results comparing the newly derived algorithms with the Crank–Nicolson scheme and certain ADI methods are presented.

1. Introduction

The numerical approximation of parabolic problems using time-splitting procedures has been a wide field of research since the pioneering works of Douglas, Peaceman and Rachford in the decade of the 1950s (cf. [6,10,11,26]). In such works, they introduced the so-called alternating direction implicit (ADI) methods by noting that, in a d-dimensional spatial domain Ω, the diffusion operator $-\nabla\cdot(a\nabla)$ can be expressed as the sum of d one-dimensional operators $\{\partial_k(a\partial_k)\}_{k=1,2,\ldots,d}$, a being a uniformly positive function on $\overline{\Omega}$ and $\partial_k = \partial/\partial x_k$. Using this idea, multidimensional parabolic problems can be solved as a sequence of one-dimensional problems, each formulated for one of the spatial variables under consideration. Any time-stepping procedure based on a component-wise splitting of this kind is called a locally one-dimensional method (see, e.g., [15]). For an extensive study of ADI and related time-splitting methods, we refer to the monographs [13,15,20].

Significantly, an ADI method can be viewed as a perturbation of some underlying implicit scheme, such as the backward Euler or the Crank–Nicolson method. In this setting, the ADI method may be formulated as the corresponding implicit scheme plus a perturbation term called the splitting error. In general, this splitting error is of the same –or higher– order in the time step τ as the truncation error associated with the underlying unsplit method. As a result, the asymptotic rate of convergence for both the ADI and its underlying method should be the same; in practice, however, the actual errors associated with the former are much larger than those associated with the latter. This fact is due to the presence of the splitting error and, typically, it is considered to be the main drawback of time-splitting methods. In order to reduce the size of such an error, Douglas and Kim introduced in [9] the so-called alternating direction method with improved initialization (AD-II)

* Corresponding author.
E-mail addresses: andres.arraras@unavarra.es (A. Arrarás), laura.portero@unavarra.es (L. Portero).

http://dx.doi.org/10.1016/j.amc.2015.03.073
0096-3003/

(sometimes referred to as the modified alternating direction (AD-M) method, see [1]). Essentially, they proposed to add a correction term to the right-hand side of the original ADI scheme with the aim of reducing the splitting error from $\mathcal{O}(\tau^2)$ to $\mathcal{O}(\tau^3)$. This idea was used in [1] to formulate improved ADI methods for regular and mixed finite element discretizations, and further studied in [19] to derive linear multistep methods by the approximate factorization technique.

In this paper, we extend the improved initialization procedure of Douglas and Kim to the case of domain decomposition-based splittings. This kind of splittings was first introduced in [35,36] to obtain the so-called regionally-additive schemes, and has been subsequently studied in [22,29,37] for the solution of evolutionary problems. In the context of linear and semi-linear parabolic equations, it has been successfully used in combination with various spatial discretization techniques, such as mimetic finite differences (cf. [3,28]), mixed finite elements (cf. [2]) or multipoint flux approximation methods (cf. [5]). Some additional results regarding nonlinear parabolic equations can be found in [4,27]. The monographs [21,31] show an overview of some recent contributions to the topic.

The key to the efficiency of domain decomposition splitting methods lies in reducing the system matrix to a collection of uncoupled submatrices of lower dimension. As compared to classical overlapping domain decomposition algorithms (cf. [30]), this approach does not involve any Schwarz iteration procedure, thus reducing the computational cost of the overall solution process. In addition, it overcomes two typical limitations of alternating direction splittings, namely: (a) their need to deal with rectangular or hexahedral spatial grids (in two- or three-dimensional problems, respectively); and (b) their difficulty to handle mixed derivative terms. In this respect, although several ADI methods have been specifically designed along the years to overcome this latter constraint (see, e.g., [8,23,24] or, more recently, [16–18]), no AD-II scheme with this property has been developed so far.

In order to introduce the improved time-splitting procedures, we consider a parabolic initial-boundary value problem of the form

$$u_t - \nabla \cdot (a\nabla u) + cu = f, \quad \mathbf{x} \in \Omega, \quad 0 < t \leqslant T, \tag{1a}$$

$$u = 0, \quad \mathbf{x} \in \partial\Omega, \quad 0 < t \leqslant T, \tag{1b}$$

$$u = u_0, \quad \mathbf{x} \in \Omega, \quad t = 0, \tag{1c}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded open domain with boundary $\partial\Omega$, $a(\mathbf{x}) \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix function, with elements $\{a_{ij}(\mathbf{x})\}_{i,j=1,2}$, $c(\mathbf{x})$ is a uniformly positive function on $\overline{\Omega}$, and the subscript t denotes partial differentiation with respect to time. The entries of $a(\mathbf{x})$ and the terms $c(\mathbf{x}), f(\mathbf{x},t)$ and $u_0(\mathbf{x})$ are assumed to be sufficiently smooth. In the sequel, we denote by $Au = -\nabla \cdot (a\nabla u) + cu$ the elliptic operator applied to the exact solution $u(\mathbf{x},t)$. For simplicity in the exposition, we consider homogeneous Dirichlet boundary conditions, although more general conditions can also be handled.

For the sake of completeness, it must be mentioned that the solution of problem (1) can also be approximated with a high order of accuracy by means of unsplit methods. Among the time-stepping schemes that have been recently designed for this purpose, it is worth referring to the so-called ADER (Arbitrary DERivative in space and time) approach. This method was initially developed to provide high-order approximations to the solution of hyperbolic conservation laws (see, e.g., [12,32]), and was subsequently formulated for nonlinear reaction–diffusion problems (cf. [33]). More recently, it has been extended in [14] to nonlinear systems of advection–diffusion–reaction equations involving stiff source terms. Some novel contributions to the topic can be found in [25,34].

The rest of the paper is organized as follows. In Section 2, we formulate some classical unsplit implicit methods for the numerical solution of problem (1). Two time-splitting strategies are described in Section 3: on one hand, the well-known Douglas (cf. [7]) and Douglas–Rachford (cf. [11]) alternating direction methods; on the other, a non-iterative domain decomposition splitting technique based on a family of partition of unity functions. The splitting error associated with the preceding schemes is studied in Section 4. In this section, we also introduce a correction procedure to reduce the size of such an error. Numerical experiments comparing the previous time-splitting schemes with the Crank–Nicolson method are reported in Section 5. The paper ends with some concluding remarks summarized in Section 6.

2. Classical implicit methods

Let us consider a suitable partition $\mathcal{T}_h$ of the spatial domain Ω, where h denotes the maximal grid spacing. In principle, $\mathcal{T}_h$ is an unstructured grid composed of either triangular or quadrilateral elements. We will see in the next section that, if an ADI method is used for solving (1), $\mathcal{T}_h$ must be assumed to be a rectangular grid. Domain decomposition splittings, by contrast, will remain valid for unstructured partitions. On the other hand, let us consider a constant time step $\tau > 0$ and define the discrete times $t_n = n\tau$, for $n = 0, 1, \ldots, N_T + 1$, with $N_T = [T/\tau] - 1$.

In this framework, let A_h be a symmetric positive definite matrix obtained from finite difference or finite element discretization of the elliptic operator A with order of accuracy $\mathcal{O}(h^s)$. Then, if U_h^n denotes the fully discrete solution at time t_n, the classical implicit time-stepping schemes can be written together, for $n = 0, 1, \ldots, N_T$, in the form

$$\frac{U_h^{n+1} - U_h^n}{\tau} + A_h(\theta U_h^{n+1} + (1-\theta)U_h^n) = F_h^{n+\theta}, \tag{2}$$