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Comparison of discontinuous Galerkin time integration schemes for the solution of flow problems with deformable domains

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ABSTRACT

The computational efficiency of numerical solvers is known to strongly depend on the chosen time integration scheme. Thus, when solving viscous flow problems on time-varying domains, an efficient and reliable solver is one of the prerequisites for a successful solution. For this reason, we provide a comparison of two time integration schemes in terms of numerical error and CPU time usage.

The contribution of the paper lies in the mutual comparison of two different approaches for time integration of the compressible Navier–Stokes equations in arbitrary Lagrangian–Eulerian (ALE) description that are spatially discretised by the discontinuous Galerkin finite element method. The computational ability of implicit Crank–Nicolson scheme and explicit three-step Runge–Kutta scheme, computational performance of which is improved by the local time-stepping technique, are tested on two flow problems: the propagation of an isentropic vortex, which has a known analytical solution, and the viscous flow around an oscillating NACA 0012 airfoil. All numerical simulations are carried out on unstructured triangular meshes by using a continuous mapping between the reference and time-varying domains.

1. Introduction

During the last decade, the discontinuous Galerkin (DG) finite element method, first proposed by Reed and Hill [1], has become very popular for the solution of various problems from the field of fluid mechanics [2–4], electrodynamics or electromagnetism [5–7] and plasma physics [8]. The reasons behind the popularity can be attributed to the method's ability to achieve high order of spatial accuracy combined with low artificial damping, robustness and overall stability, thus, making it an ideal method for the simulation of laminar and turbulent flows with complex vortical structures. Although able to produce stable and high-order accurate solutions on fully unstructured meshes, the DG method is associated with a large number of unknowns, which with increasing problem size can substantially increase the computational demands. For this reason, the choice of a time integration method is of crucial importance [9,10].

Most often, the time discretisation is carried out with an explicit time integration scheme that represents the simplest choice. It is, however, well known that the computational efficiency of explicit schemes is directly affected by the maximal

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size of the global time-step restricted by the CFL condition of stability. This restriction becomes particularly notable in computational meshes with large variations in element size that are very common in computational practice. For example, when a local mesh refinement is needed due to the presence of sharp corners in the geometry or due to the presence of shock waves. In this case, the smallest element can significantly reduce the size of the global time step and, thus, hamper the overall computational efficiency. To overcome the disadvantage, a local time-stepping (LTS) technique can be used, see, e.g., [7]. As the name suggests, the LTS method utilises the principle of local time steps that are computed for each element independently. In this way, a massive reduction of CPU time can be achieved without the need to significantly alter the computational algorithm developed for an explicit scheme.

Another way, how to overcame the restriction imposed by the CFL condition, is to introduce an implicit time integration method with a large interval of stability. Although this approach may seem straightforward, it is a fact that the application of implicit schemes gives rise to several problems. Among them, it is possible to mention the need to solve a system of non-linear algebraic equations during every time iteration, which in large size problems may negatively affect the CPU time. It is also not insignificant that implicit schemes are generally difficult to program and cannot be easily parallelised as the explicit ones. Moreover, meshes with elements of highly disparate size are known to lead to ill-conditioned problems with implicit schemes.

By taking into consideration all the advantages and disadvantages of explicit and implicit schemes, a question naturally arises, whether the modification of explicit schemes in the form of the LTS method can provide higher or at least comparable computational performance than their implicit counterparts. For unsteady 3D viscous flow simulations in non-deforming domains, an answer to this question can be found in [9], where the authors compared various Jacobian free implicit methods with selected explicit ones. On the basis of the presented results, the authors concluded that explicit schemes can achieve approximately the same computational performance as the implicit ones when the LTS technique is applied, as well.

By following up the work of Birken et al. [9], we try to find an answer to the question above by solving unsteady flow problems in time-varying domains, where the numerical solution is also dependent on the mapping procedure between the fixed reference configuration and the real domain. To determine the impact of time integration on CPU time usage and numerical error, we consider two well-known explicit and implicit time integration schemes – the explicit three-step Runge–Kutta method, performance of which is improved by the local time-stepping method, and the implicit Crank–Nicolson method.

The structure of the paper is outlined as follows: First, the equations governing the viscous flow in a time-varying domain are introduced and discretised in the sense of the DG method. Next, the time integration with the explicit LTS and implicit schemes is carried out, followed by a description of the mapping procedure necessary for the solution of flow problems on moving domains. In the next section, the computational ability of the developed explicit and implicit solvers is first verified for an inviscid model problem consisting of a moving isentropic vortex [11] and then used for the numerical simulation of viscous flow around an oscillating NACA 0012 airfoil, motion of which is prescribed by a time-dependent function. Finally, we summarise our work, give some concluding remarks and provide a brief outline of future work.

2. Mathematical model and DG discretisation

2.1. Governing equations

The mathematical model describing the compressible viscous flow in a two-dimensional time-varying computational domain consists of the non-linear conservative system of the Navier–Stokes equations in ALE form. The equations written in the compact vector form are given as

$$\frac{D^A \boldsymbol{w}}{Dt} + \sum_{s=1}^{2}\left(\frac{\partial}{\partial x_s}\boldsymbol{f}_s(\boldsymbol{w}) - U_s\frac{\partial \boldsymbol{w}}{\partial x_s}\right) = \sum_{s=1}^{2}\frac{\partial}{\partial x_s}\boldsymbol{f}_s^v(\boldsymbol{w}, \nabla \boldsymbol{w}), \tag{1}$$

where $\frac{D^A}{Dt}$ denotes the so-called ALE time derivative, $[x_1, x_2] = \boldsymbol{x} \in \Omega(t) \subset \mathbb{R}^2,\ t \in [0, \mathcal{T}],\ \boldsymbol{w}(\boldsymbol{x}, t) = [\varrho, \varrho u_1, \varrho u_2, E]^T$ is the vector of conservative variables, ϱ is the density, $\boldsymbol{u} = [u_1, u_2]^T$ is the velocity vector, E is the total energy per unit volume, $\boldsymbol{f}_s(\boldsymbol{w}) = [\varrho u_s, \varrho u_s u_1 + \delta_{s1}p, \varrho u_s u_2 + \delta_{s2}p, (E+p)u_s]^T$ and $\boldsymbol{f}_s^v(\boldsymbol{w}) = [0, \tau_{1s}, \tau_{2s}, u_1\tau_{1s} + u_2\tau_{2s} + k\partial T/\partial x_s]^T$ are the inviscid and viscous fluxes, respectively. The stress tensor τ_{ij} is defined as

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right), \quad i,j = 1,2, \tag{2}$$

where μ is the dynamic viscosity and δ_{ij} is the Kronecker delta. By U_s we denote the components of the domain velocity. The system of governing equations (1) is completed by the constitutive relations for ideal gas and temperature

$$p = (\kappa - 1)\left[E - \frac{1}{2}\varrho\sum_{s=1}^{2}u_s^2\right], \quad k\frac{\partial T}{\partial x_s} = \frac{\kappa}{\kappa - 1}\frac{\mu}{\mathrm{Pr}}\frac{\partial}{\partial x_s}\left(\frac{p}{\varrho}\right), \tag{3}$$

where p is the pressure, $\kappa = 1.4$ is the adiabatic index and $\mathrm{Pr} = 0.72$ is the Prandtl number.