Applied Mathematics and Computation 267 (2015) 341–355

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A framework of verified eigenvalue bounds for self-adjoint differential operators

Xuefeng Liu

Graduate School of Science and Technology, Niigata University, 8050 Ikarashi 2-no-cho, Nishi-ku, Niigata City, Niigata 950-2181, Japan

ARTICLE INFO

Keywords:
Self-adjoint differential operator
Eigenvalue bounds
Non-conforming finite element method
Quantitative error estimation
Verified computation

ABSTRACT

For eigenvalue problems of self-adjoint differential operators, a universal framework is proposed to give explicit lower and upper bounds for the eigenvalues. In the case of the Laplacian operator, by applying Crouzeix–Raviart finite elements, an efficient algorithm is developed to bound the eigenvalues for the Laplacian defined in 1D, 2D and 3D spaces. Moreover, for nonconvex domains, for which case there may exist singularities of eigenfunctions around re-entrant corners, the proposed algorithm can easily provide eigenvalue bounds. By further adopting the interval arithmetic, the explicit eigenvalue bounds from numerical computations can be mathematically correct.

1. Introduction

The eigenvalue problem plays an important role in both natural and engineering sciences. In this paper, we consider the class of self-adjoint eigenvalue problems, the eigenvalues of which are real numbers, and propose a universal framework to give lower and upper bounds for eigenvalues.

For a long time, the numerical analysis for eigenvalue problems, for example, of Laplacian eigenvalues, has been well documented in literature. Most classical research focuses on the qualitative analysis of numerical schemes, such as convergence order. But, quantitative analysis, for example, explicit eigenvalue bounds, has not drawn much interest from researchers.

Recently, explicit eigenvalue bounds have become more indispensable, especially in adaptive computing of the finite element method (FEM) and in the computer-assisted proof for nonlinear differential equations. For example, a good indicator for the error of approximate solutions requires the explicit error estimation for various interpolation operators. The estimation of error constants is reduced to eigenvalue problems of Laplace and biharmonic operators; see, [10,13]. In addition, verifying the solution for nonlinear differential equations requires eigenvalue bounds of the controlling differential operators; see, e.g., [17,19,22].

Generally, we can easily obtain upper bounds for eigenvalues by using Rayleigh–Ritz's method, but lower eigenvalue bounds remain difficult to find. Theoretical analysis of eigenvalue bounds, which is independent of numerical scheme selection, includes the early work of Kato, Weinstein and Stenger, Lehmann, Beattie and Goerisch, Behnke and Goerisch, Goerisch [9,23,11,1,4,7]. These theories provide nice eigenvalue bounds, assuming there are rough a priori bounds for the eigenvalues. A good choice to provide the necessary a priori eigenvalue bounds is the homotopy method proposed by Plum [18], which considers the connection between the base problem—with a known spectrum—to the objective problem. With a domain transformation, this method can even deal with the domain of general shapes. However, to apply the homotopy method in solving practical problems, we need case-by-case efforts in setting up the homotopy process.

E-mail address: xfliu@math.sc.niigata-u.ac.jp

http://dx.doi.org/10.1016/j.amc.2015.03.048

In practical computation, such methods as the FEM, finite difference method, and intermediate method, can result in good eigenvalue approximation. However, most of these methods have difficulties in dealing with domains of general shapes if rigorous eigenvalue bounds are wanted, and the indices of eigenvalues are not easy to verify; see [15] for a review of such methods and the reference therein. In Liu and Oishi [15], by applying the hypercircle equation technique, the authors developed an algorithm that can provide guaranteed lower and upper eigenvalue bounds for the Laplacian. By inheriting the advantage of FEMs, such an algorithm can naturally deal with eigenvalue problem over domains of arbitrary shape.

In this paper, we extend the algorithm of Liu and Oishi [15] to more general cases defined by abstract bilinear forms. Moreover, the proposed method enables the utility of non-conforming FEMs. For non-conforming finite elements, the numerical results themselves can be lower bounds for eigenvalues if the mesh size is small enough from the asymptotic analysis (see, e.g., [24,16]). But the necessary small enough condition usually cannot be verified explicitly. In our proposed algorithm, based on the computation results of non-conforming FEMs, guaranteed lower eigenvalue bounds are possible, even for a very raw mesh. The proposed algorithm can deal with the Laplace and Biharmonic eigenvalue problems. In this paper, we focus on the Laplacian eigenvalue problems.

For the eigenvalue problem of Laplacian, the Crouzeix–Raviart finite element is adopted to give lower eigenvalue bounds (see details in Section 3): Let λ_k be the kth eigenvalue and $\lambda_{h,k}$ the kth approximate eigenvalue. A lower bound of λ_k is given as

$$\frac{\lambda_{h,k}}{1+C_h^2\lambda_{h,k}} \leqslant \lambda_k,$$

where C_h is a constant related to error estimation for the Crouzeix–Raviart interpolation Π_h; see definition in Section 3.1. Let the diameter of an element K be h. The constant C_h is the one to make the following estimation hold:

$$\|u-\Pi_h u\|_{0,K} \leqslant C_h |u-\Pi_h u|_{1,K}.$$

Here,

- $C_h = h/\pi$ when K is an interval in $\mathbf{R}^1$, which is an already known result;
- $C_h = 0.1893h$ for a triangle element K in $\mathbf{R}^2$;
- $C_h = 0.3804h$ for a tetrahedron element K in $\mathbf{R}^3$.

Moreover, the selection of C_h for $\mathbf{R}^1$ is optimal and the value $C_h = 0.1893h$ for $\mathbf{R}^2$ is very near to optimal.

When this research was almost finished, we found independent results of Carstensen and Gallistl [5,6], which also use non-conforming FEMs to give lower eigenvalue bounds, but a separation condition is needed. As explained in Remark 3.1, the separation condition is in fact not needed. Also, our results give better estimation of the constant C_h for eigenvalue problems of the Laplacian in the 2D case.

The remainder of this paper is organized as follows: In Section 2, we introduce the eigenvalue problem defined in an abstract form along with the main theorem that provides lower eigenvalue bounds. In Section 3, the eigenvalue problem of the Laplacian is considered in $\mathbf{R}^m (m=1,2,3)$. In Section 4, an optimal estimation of the error constant C_h for 2D case is given. In Section 5, the computation results are presented. Finally, in Section 6, we state our conclusions and discuss the scope for future work.

2. Abstractly defined eigenvalue problems and lower eigenvalue bounds

Let Ω be a domain of $\mathbf{R}^m$ $(m=1,2,3)$. We show the assumptions for function spaces to be used in the main theorem on eigenvalue bounds.

A1 V is a Hilbert space of real function on Ω with the inner product $M(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|_M := \sqrt{M(\cdot,\cdot)}$.

A2 $N(\cdot,\cdot)$ is another inner product of V. The corresponding norm $\|\cdot\|_N := \sqrt{N(\cdot,\cdot)}$ is compact for V with respect to $\|\cdot\|_M$, i.e., every sequence in V which is bounded in $\|\cdot\|_M$ has a subsequence which is Cauchy in $\|\cdot\|_N$.

To deal with conforming or non-conforming finite element spaces in eigenvalue evaluations, we further take the following assumptions.

A3 V^h is a finite dimensional space of real function over Ω, $\mathrm{Dim}(V^h)=n$ (the value of n is fixed). Notice that V^h may not be a subspace of V. Define $V(h) := V + V^h = \{v + v_h | v \in V, v_h \in V^h\}$.

A4 Bilinear forms $M_h(\cdot,\cdot)$ and $N_h(\cdot,\cdot)$ on $V(h)$ are extension of $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ to $V(h)$ such that
- $M_h(u,v) = M(u,v)$, $N_h(u,v) = N(u,v)$ for all u, $v \in V$.
- $M_h(\cdot,\cdot)$ and $N_h(\cdot,\cdot)$ are symmetric and positive definite on $V(h)$.

The assumption A4 implies that $M_h(\cdot,\cdot)$ and $N_h(\cdot,\cdot)$ are also inner products of $V(h)$. For purpose of simplicity, the extended bilinear forms $M_h(\cdot,\cdot)$ and $N_h(\cdot,\cdot)$ are still denoted by $M(\cdot,\cdot)$ and $N(\cdot,\cdot)$ and the corresponding norms are denoted by $\|\cdot\|_M$ and