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Numerical simulation of the interaction between a nonlinear elastic structure and compressible flow by the discontinuous Galerkin method

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A B S T R A C T

This paper is concerned with the numerical simulation of the interaction of compressible viscous flow with a nonlinear elastic structure. The flow is described by the compressible Navier–Stokes equations written in the arbitrary Lagrangian–Eulerian (ALE) form. For the elastic deformation the St. Venant–Kirchhoff model is used. In the space discretization the discontinuous Galerkin finite element method (DGM) is applied both for the flow problem in a time-dependent domain and for the dynamic nonlinear elasticity system. We show that the DGM is applicable to the discretization of both problems. As a new result we particularly present the application of the DGM to the discretization of the dynamic nonlinear elasticity problem and the DGM solution of the fluid–structure interaction (FSI). The applicability of the developed technique is demonstrated by several numerical experiments. The main novelty of the paper is the application of the DGM to the FSI problem using the model of compressible flow coupled with nonlinear elasto-dynamic system.

1. Introduction

Fluid–structure interaction problems are solved in various aerospace, civil and mechanical engineering applications for many years, see e.g. monographs [13,21] and the proceedings [5,6], where various aspects of FSI modeling and simulation are treated. To the best of our knowledge, mostly the model of incompressible fluid is used and the numerical techniques for the FSI simulation are based on the application of the finite volume method or conforming finite elements. Recently the methods for solution of such problems have also been quickly developed in the field of biomechanics. We are particularly interested in biomechanics of voice, where the methods of numerical simulation of human vocal folds self-oscillation are currently in an intensive development. These self-oscillations, which originate in the interaction of airflow coming from the human lungs with the compliant biological tissue of the vocal folds, produce primary sound enabling voicing (phonation, speech, singing), see e.g. [20]. One of the latest papers published by Tian et al. [23] considers an interaction of incompressible viscous airflow described by the Navier–Stokes equations with nonlinear elastic structure of the human vocal folds. For large vibration amplitudes the authors are solving the problem by using the so-called immersed boundary method for flow and

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finite element method for the viscoelastic tissue with large shape changes of the biological structure modeling the vocal folds. For further models and aspects of the numerical simulation of vocal folds vibrations we refer to the works [1,19,25,26].

A nonlinear benchmark problem on FSI was proposed in [24], where a beam vibrating in incompressible laminar flow is modeled as a linear elastic structure with the geometrical nonlinearity. However, until now not many papers on FSI deal with dynamics of a nonlinear continuum. Even very few static problems using the model of nonlinear continuum are analyzed, see e.g. [4]. Especially, models of biological viscoelastic tissues subjected to large deformations are still unresolved.

Here we are concerned with the development of a method for the numerical simulation of an interaction of a 2D nonlinear elastic structure with 2D low Mach number compressible viscous flow. In [15,18], the dynamic linear elasticity initial-boundary value problem is discretized by the conforming finite element method in space and by the Newmark method in time. The compressible Navier–Stokes equations are formulated in the arbitrary Lagrangian–Eulerian (ALE) form in order to take the time-dependence of the flow domain into account and solved by the discontinuous Galerkin method (DGM) using piecewise polynomial approximations of the exact solution on a finite element mesh without the requirement of the continuity on interfaces between neighboring elements. Several works (e.g. [2,11,14]) prove that this method is suitable for numerical approximations of nonlinear convection–diffusion problems and compressible flow, when the solutions contain discontinuities and/or internal and boundary layers.

In this paper we shall present the numerical method based on the discontinuous Galerkin method for flow described by the compressible Navier–Stokes equations coupled with the DGM solution for the elastic structure taking large deformations into account. For the description of the deformation of the elastic structure the nonlinear St. Venant–Kirchhoff model is used, see e.g. [7].

The main emphasis is paid to the numerical solution of the nonlinear elasticity problem by the DGM in space and both the backward-difference formula (BDF) and the DGM in time, treated in Section 2. The developed method represents the novelty in the simulation of nonlinear elasticity.

In order to demonstrate the applicability of the developed method, in Section 3 we present several numerical experiments for the simulation of the deformation of an elastic beam, which is inspired by the benchmark described by Turek and Hron, see [24].

Finally, the solution of the coupled FSI problem is described. The coupling is realized via transmission conditions, which are implemented in the numerical process with the aid of a strong coupling algorithm.

The developed method can be applied to problems of biomechanics, aerodynamics and aviation. Our work is particularly motivated by the simulation of airflow in a simplified model of human airways created by the trachea, a glottal region with vibrating vocal folds and the vocal tract channel. The presented numerical experiment is motivated by the phonation process. The results demonstrate the applicability of the developed method and its robustness with respect to low Mach number flows and a suitable nonlinear elastic structure.

2. Nonlinear elasticity

2.1. Notation

We consider an elastic structure with *reference (undeformed) configuration* represented by an open set $\Omega^b \subset \mathbb{R}^2$. It is deformed via a mapping $\boldsymbol{\varphi}$ taking a material point $\boldsymbol{x}$ to a point $\boldsymbol{\varphi}(\boldsymbol{x})$ in space. The deformation mapping can be expressed by the use of a displacement function $\boldsymbol{u}$ as $\boldsymbol{\varphi}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$. By $\boldsymbol{F} := \nabla\boldsymbol{\varphi}(\boldsymbol{x})$ we denote the *deformation gradient*, i.e. $\boldsymbol{F}$ is the Jacobian matrix of the deformation mapping $\boldsymbol{\varphi}$. Further, $J = \det \boldsymbol{F} > 0$ denotes the Jacobian of the deformation. By δ_{ij} we denote the Kronecker symbol and set $\boldsymbol{I} = (\delta_{ij})_{i,j=1}^2$. If we write $\boldsymbol{x} = (x_1, x_2)^T$ and $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))^T$, then the deformation gradient can be expressed as

$$\boldsymbol{F} = (F_{ij})_{i,j=1}^2, \quad F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}, \quad i,\ j = 1,2. \tag{1}$$

2.1.1. Energy

The energy that is stored in the deformed body is referred to as *strain energy* (see [7]). We assume that the strain energy is determined by the deformation mapping of the reference configuration and therefore we use the notation $E(\boldsymbol{\varphi}(\boldsymbol{x}))$. We shall only deal with the so-called *hyperelastic materials* where the *first Piola–Kirchhoff stress tensor* $\boldsymbol{P}$ has a potential $W = W(\boldsymbol{F})$ called *stored energy density*, i.e.

$$\boldsymbol{P}(\boldsymbol{F}) = \frac{\partial W(\boldsymbol{F})}{\partial \boldsymbol{F}}. \tag{2}$$

We obtain the strain energy of the deformed body by integrating the energy density function over the entire reference domain Ω^b:

$$E(\boldsymbol{\varphi}) = \int_{\Omega^b} W(\nabla\boldsymbol{\varphi}(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}. \tag{3}$$