Applied Mathematics and Computation 267 (2015) 397–408

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Tight bounds on angle sums of nonobtuse simplices

Jan Brandts [a,*], Apo Cihangir [a], Michal Křížek [b]

[a] Korteweg-de Vries Institute for Mathematics, University of Amsterdam, P.O. Box 94248, 1090 GE Amsterdam, The Netherlands
[b] Institute of Mathematics, Academy of Sciences, Žitna 25, CZ-115 67 Prague 1, Czech Republic

ARTICLE INFO

Keywords:
Simplex
Nonobtuse simplex
Angle sums
Spherical geometry
Spherically convex function
Polar simplex

ABSTRACT

It is widely known that the sum of the angles of a triangle equals two right angles. Far less known are the answers to similar questions for tetrahedra and higher dimensional simplices. In this paper we review some of these less known results, and look at them from a different point of view. Then we continue to derive tight bounds on the dihedral angle sums for the subclass of nonobtuse simplices. All the dihedral angles of such simplices are less than or equal to right. They have several important applications (Brandts et al., 2009). The main conclusion is that when the spatial dimension n is even, the range of dihedral angle sums of nonobtuse simplices is n times smaller than the corresponding range for arbitrary simplices. When n is odd, it is $n-1$ times smaller.

1. Introduction

In his 1952 paper [5], Gaddum addressed the relative obscurity of facts about the sums of *dihedral angles* of simplices, and particularly of tetrahedra. The dihedral angle between two facets of an n-simplex is the angle α_{ij} supplementary to the angle γ_{ij} between two *outward normals* q_i and q_j to those facets. For a triangle this is illustrated in the left of Fig. 1. Note that the angle γ_{ij} equals the *geodesic distance* between *unit length* outward normals, which are points on the $(n-1)$-sphere $\mathbb{S}^{n-1}$. This is illustrated in the right of Fig. 1, where depict four outward normals of an invisible tetrahedron, and their intersections with $\mathbb{S}^2$.

Write Σ_n for the sum of all dihedral angles of an n-simplex S, and Γ_n for the sum of the geodesic distances on $\mathbb{S}^{n-1}$ between each pair of the $n+1$ unit length outward normals to the facets of S. Then obviously,

$$\Sigma_n + \Gamma_n = \binom{n+1}{2}\pi. \qquad (1)$$

In [5], Gaddum used (1) to prove that the sum Σ_3 of the six dihedral angles between the triangular faces of a tetrahedron can take any value greater than 2π and less than 3π. Indeed, he considered the extrema of the sum Γ_3 for four *global* points on the unit sphere $\mathbb{S}^2$, which means that they do not all lie on the same closed hemisphere. It is not difficult to show that $n+1$ points on $\mathbb{S}^{n-1}$ are global if and only if they are unit length outward pointing normals to the facets of an n-simplex.

In the follow-up paper [6] from 1956, Gaddum derived similar bounds for the sum Σ_n of the dihedral angles of an arbitrary n-simplex. We will discuss these bounds in Section 2 in detail, presenting them in a different form than in the original

* Corresponding author.
E-mail addresses: J.H.Brandts@uva.nl (J. Brandts), A.Cihangir@uva.nl (A. Cihangir), krizek@math.cas.cz (M. Křížek).

http://dx.doi.org/10.1016/j.amc.2015.02.035
0096-3003/

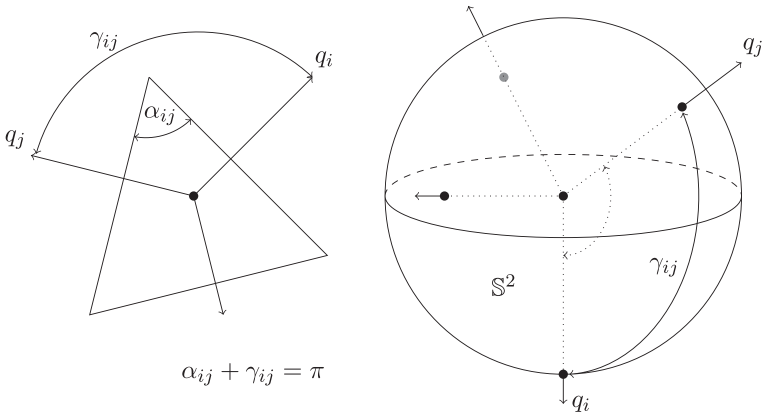

Fig. 1. Left: The dihedral angle and the angle between outward normals to the facets add up to π. Right: the angle between the normals seen as geodesic distance on the sphere.

papers, while explicitly constructing simplices whose dihedral angle sums tend to the bounds. In Section 3 we will present new results, which are valid for the important subclass of nonobtuse simplices. All the dihedral angles of such simplices are less than or equal to right. They were the object of study in [1–3,7,8]. Proofs are given in the subsequent sections. These proofs are based on convex optimization on $\mathbb{S}^{n-1}$, as opposed to the proofs of Gaddum in [6] in which higher dimensional generalizations of Girard's Theorem of *spherical excess* were derived. Unfortunately, it seems that neither Gaddum's approach can be applied to derive tight dihedral angle bounds for nonobtuse simplices, nor that our proofs can be generalized as to include Gaddum's bounds for arbitrary simplices.

2. Detailed discussion of Gaddum's dihedral angle bounds

Using different but equivalent expressions as Gaddum, we will first rephrase his central result. Unlike in Section 3 to come, we express angles as multiples of π.

Theorem 2.1 (Gaddum [6]). *For $n \geqslant 3$, the dihedral angle sum Σ_n of an n-simplex satisfies*

$$\lambda(n)\pi < \Sigma_n < \mu(n)\pi, \tag{2}$$

where

$$\lambda(n) = \binom{n+1}{2} - \left\lfloor \frac{n+1}{2} \right\rfloor \left\lceil \frac{n+1}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \quad \text{and} \quad \mu(n) = \binom{n+1}{2} - n = \binom{n}{2}. \tag{3}$$

Moreover, for each $\alpha \in (\lambda(n)\pi, \mu(n)\pi)$ there is a simplex with dihedral angle sum equal to α.

Unlike [6], we have explicitly written $\lambda(n)$ and $\mu(n)$ in (3) as a binomial coefficient, which equals the total number of dihedral angles, minus a correction. In the case of $\mu(n)$ and in view of (2), this correction shows that not all dihedral angles of a simplex can be arbitrarily close to π. Indeed, according to a theorem by Fiedler [4] from 1957, each n-simplex has at least n acute (smaller than right) dihedral angles. Thus, the expression for $\mu(n)$ in (2) suggests that the upper bound may be approached by a simplex with n dihedral angles tending to zero, whereas its remaining dihedral angles approach π. On the other hand, the expression for $\lambda(n)$ in (3) shows, in view of the lower bound in (2), that not all dihedral angles of a simplex can tend to zero. It also suggests that the lower bound in (2) might be approached by a simplex that has $\lfloor \frac{n+1}{2} \rfloor \lceil \frac{n+1}{2} \rceil$ angles tending to zero, whereas the remaining $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ tend to π. We will proceed to show that both the above observations on $\mu(n)$ and $\lambda(n)$ are indeed correct.

2.1. Informal discussion of the three-dimensional case

The tetrahedron on the right in Fig. 2 has a vertex very near to a point in the interior of the opposite facet. Hence, all four facets lie almost in the same plane. One faces downward, and three face almost upward, which is symbolized with the diagram on the right in Fig. 2. The three upward facing facets make very small dihedral angles with the facet facing downward, whereas the dihedral angle between each of the three pairs of distinct upward facing facets approaches π. This leads to a total dihedral angle sum that approaches 3π as the height of the vertex above the ground facet tends to zero. It confirms our claim that the upper bound in (2) can be approached by tetrahedra with dihedral angles tending to π whereas the remaining three tend to zero.