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A periodic basis system of the smooth approximation space

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ABSTRACT

The paper is primarily devoted to the problem of smooth interpolation of data in 1D. In addition to the exact interpolation of data at nodes, we are also concerned with the smoothness of the interpolating curve and its derivatives.

The interpolating curve is defined as the solution of a variational problem with constraints. The system of functions $\exp(-ikx)$, k being an integer, is taken for the basis of the space where we measure the smoothness of the result. It also generates the functions used for the interpolation itself. Choosing different norms when measuring the smoothness, we arrive at different interpolating functions. We also mention the problem of smooth curve fitting (data smoothing).

We discuss the proper choice of different norms for this way of approximation and present the results of several 1D numerical examples that show the advantages and drawbacks of smooth interpolation.

1. Introduction

Smooth approximation [11] is an approach to data interpolating or data fitting that employs the variational formulation of the problem in a normed space with constraints representing the approximation conditions. A possible criterion is to minimize the integral of the squared magnitude of the approximating function. A more sophisticated criterion is then to minimize, with some weights chosen, the integrals of the squared magnitude of some (or possibly all) derivatives of a sufficiently smooth approximating function. In the paper, we are concerned with the exact interpolation of the data at nodes and, at the same time, with the smoothness of the interpolating curve and its derivatives.

Smooth approximation has numerous applications as measurements of the values of a continuous function of one, two, or three independent variables are carried out in many branches of science and technology. We always get a finite number of function values measured at a finite number of nodes but we are interested also in values corresponding to other points. Apparently, except for the fixed constraints to be satisfied, the formulation of the problem of smooth approximation can vary and give the resulting interpolant of different behavior. The cubic spline interpolation is also known to be the approximation of this kind.

We are mostly interested in the case of a single independent variable in the paper. Assuming the approach of [4,9,11], we introduce the problem to be solved and the tools necessary to this aim in Section 3. We also present the general existence theorems for smooth interpolation and smooth curve fitting [9]. We are concerned with the use of basis system $\exp(ikx)$ of exponential functions of pure imaginary argument for 1D, 2D, and 3D smooth approximation problems in Sections 4 and 5. We investigate some of its properties suitable for measuring the smoothness of the approximation and for generating the functions used for the practical interpolation. In the next section, we show results of numerical experiments to compare

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the classical interpolation formulae and various basis systems for the smooth approximation. We finally discuss the results presented that illustrate some properties of smooth approximation.

2. Problem of data approximation

Let us have a finite number N of (complex, in general) measured (sampled) values $f_1, f_2, \ldots, f_N \in C$ obtained at N nodes $X_1, X_2, \ldots, X_N \in R^n$. The nodes are assumed to be mutually distinct. We are usually interested also in the intermediate values corresponding to other points in some domain. Assume that $f_j = f(X_j)$ are measured values of some continuous function f while z is an approximating function to be constructed. The dimension n of the independent variable can be arbitrary. For the sake of simplicity we put $n = 1$ except for Section 5 and assume that $X_1, X_2, \ldots, X_N \in \Omega$, where either $\Omega = [a, b]$ is a finite interval or $\Omega = (-\infty, \infty)$.

2.1. Data interpolation

The interpolating function z is constructed to fulfil the interpolation conditions

$$z(X_j) = f(X_j), \quad j = 1, \ldots, N. \tag{1}$$

Some additional conditions can be considered, e.g. the Hermite interpolation or minimization of some functionals applied to z.

2.2. Curve fitting (data smoothing)

Instead of satisfying the interpolation equalities (1) we look for a smoothing function $\hat{z}$ that minimizes the expression

$$\sum_{j=1}^{N} w_j(\hat{z}(X_j) - f_j)\left(\hat{z}^*(X_j) - f_j^*\right), \tag{2}$$

where * denotes complex conjugation and $w_j, j = 1, \ldots, N$, are positive weights chosen (smoothing by the least squares method). Put $W = \mathrm{diag}(w_1, \ldots, w_N)$.

Additional conditions, e.g. the minimization of some further functionals applied to $\hat{z}$, can be imposed, too.

The problem of data approximation does not have a unique solution. The properties (1) or (2) of the interpolating or smoothing function are uniquely formulated by mathematical means but there are also requirements on the *subjective perception* of the behavior of the approximating curve or surface between nodes that can hardly be formalized [12].

3. Smooth approximation

We introduce an inner product space to formulate the additional constraints in the problem of smooth approximation [8,9,11]. Let $\widetilde{\mathcal{W}}$ be a linear vector space of complex valued functions g continuous together with their derivatives of all orders on the interval Ω. Let $\{B_l\}_{l=0}^{\infty}$ be a sequence of nonnegative numbers and L the smallest nonnegative integer such that $B_L > 0$ while $B_l = 0$ for $l < L$. For $g, h \in \widetilde{\mathcal{W}}$, put

$$(g, h)_L = \sum_{l=0}^{\infty} B_l \int_{\Omega} g^{(l)}(x)[h^{(l)}(x)]^* \,\mathrm{d}x, \tag{3}$$

$$|g|_L^2 = \sum_{l=0}^{\infty} B_l \int_{\Omega} |g^{(l)}(x)|^2 \,\mathrm{d}x. \tag{4}$$

If $L = 0$ (i.e. $B_0 > 0$), consider functions $g \in \widetilde{\mathcal{W}}$ such that the value of $|g|_0$ exists and is finite. Then $(g, h)_0 = (g, h)$ has the properties of *inner product* and the expression $|g|_0 = \|g\|$ is *norm* in a normed space $W_0 = \widetilde{\mathcal{W}}$.

Let $L > 0$. Consider again functions $g \in \widetilde{\mathcal{W}}$ such that the value of $|g|_L$ exists and is finite. Let $P_{L-1} \subset \widetilde{\mathcal{W}}$ be the subspace whose basis $\{\varphi_p\}$ consists of monomials

$$\varphi_p(x) = x^{p-1}, \quad p = 1, \ldots, L.$$

Then $(\varphi_p, \varphi_q)_L = 0$ and $|\varphi_p|_L = 0$ for $p, q = 1, \ldots, L$. Using (3) and (4), we construct the *quotient space* $\widetilde{\mathcal{W}}/P_{L-1}$ whose zero class is the subspace P_{L-1}. Finally, considering $(\cdot,\cdot)_L$ and $|\cdot|_L$ in every equivalence class, we see that they represent the inner product and norm in the normed space $W_L = \widetilde{\mathcal{W}}/P_{L-1}$.

W_L is the normed space where we minimize functionals and measure the smoothness of the interpolation. For an arbitrary $L \geqslant 0$, choose a *basis system* of functions $\{g_k\} \subset W_L, k = 1, 2, \ldots$, that is complete and orthogonal (in the inner product