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Solving elliptic eigenvalue problems on polygonal meshes using discontinuous Galerkin composite finite element methods

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ABSTRACT

In this paper we introduce a discontinuous Galerkin method on polygonal meshes. This method arises from the discontinuous Galerkin composite finite element method (DGFEM) for source problems on domains with micro-structures. In the context of the present paper, the flexibility of DGFEM is applied to handle polygonal meshes. We prove the a priori convergence of the method for both eigenvalues and eigenfunctions for elliptic eigenvalue problems. Numerical experiments highlighting the performance of the proposed method for problems with discontinuous coefficients and on convex and non-convex polygonal meshes are presented.

1. Introduction

In recent years people have realised the gain in flexibility coming from polygonal and polyhedral meshes, for this reason many finite element methods have been developed to accommodate such meshes. In the continuous Galerkin setting we have the composite finite element methods (CFEs) [1–5], the polygonal finite element methods (PFEMs) [6,7], the extended finite element method (XFEM) [8] and the virtual finite element method (VFEM) [9]. On the other hand in the discontinuous Galerkin (DG) setting we have the interior penalty methods on polygonal and polyhedral meshes [10], the agglomeration-based method [11–13] and the discontinuous Galerkin composite finite element methods (DGCFEMs) [14].

One clear advantage in using general polygonal/polyhedral elements is the possibility to mesh complicated shapes and even small geometrical details in the domain. In this direction both the continuous Galerkin CFE method [1–5] and the discontinuous Galerkin CFE method [15,14,16] are capable to solve problems on domains with micro-structures. It is interesting to notice that on domains without small features or micro-structures, DGCFEM [14] and the interior penalty methods on polygonal and polyhedral meshes [10] are closely related. From an a priori convergence point of view, the main difference between these two methods is the way in which degeneration of edges or faces in the mesh is treated in the theory.

In this work we would like to study the use of polygonal/polyhedral meshes for eigenvalue problems. In our opinion, this seems the natural next step since so many methods for linear/non-linear source problems are already available for such meshes. Comparing the extensions of continuous and discontinuous Galerkin methods to general polygonal/polyhedral meshes, clearly in the DG setting the extension is simpler. For this reason we adopted DG as our starting point. Among all the available DG methods, we choose to apply DGCFEM to eigenvalue problems since *hp*-adaptive schemes are already available for this method [16,15] and it should be possible in a further work to apply such technologies to eigenvalue problems on general polygonal/polyhedral meshes as well. Moreover, since DGCFEM has been developed to address problems on

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domains with micro-structures, also the present analysis can be applied to eigenvalue problems on domain with micro-structures using polygonal/polyhedral meshes.

In order to keep the analysis simple, we consider the following model problem: find the eigenpairs (λ, u) such that

$$\begin{cases} -\Delta u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, Ω is a bounded, connected polygonal domain in $\mathbb{R}^2$, with boundary $\partial\Omega$. In the rest of the paper we are going to assume that the meshes are constituted by polygonal elements.

The outline of the paper is as follows. In Section 2 we describe how the finite element space on polygonal meshes is constructed. In Section 3 we introduce the discrete version of problem (1) and the discontinuous Galerkin method. In the following Section 4 the a priori analysis is presented and in Section 5 the numerical results are presented. Finally in Section 6 some concluding remarks are collected.

2. Construction of the composite finite element spaces on polygonal meshes

In this section, we describe the construction of the CFE space on a polygonal mesh. The method is inspired by [14], where a similar construction for complicated domain with small features is presented.

The construction of the CFE space takes advantage of two meshes: the polygonal mesh $\mathcal{T}_{\mathtt{CFE}}$ and the mesh $\mathcal{T}_h$ constructed splitting the polygons in $\mathcal{T}_{\mathtt{CFE}}$ into triangles. By construction the mesh $\mathcal{T}_h$ is finer than the mesh $\mathcal{T}_{\mathtt{CFE}}$, in the sense that each element of $\mathcal{T}_h$ has a unique father element in $\mathcal{T}_{\mathtt{CFE}}$ such that the father element contains the children element.

2.1. Finite element spaces

We start defining the discontinuous Galerkin finite element space on the mesh $\mathcal{T}_h$, assuming that the polynomial degree is uniformly distributed over the mesh:

$$V(\mathcal{T}_h, p) = \{u \in L_2(\Omega) : u|_\kappa \in \mathcal{P}_p(\kappa), \forall \kappa \in \mathcal{T}_h\},$$

where $\mathcal{P}_p(\kappa)$ denotes the set of polynomials of degree at most $p \geqslant 1$ defined over the general polygon κ. The extension to variable polynomial degrees follows in a natural fashion.

In order to be able to construct the finite element space on the polygonal mesh, we have to assume that the polynomial degree p of the polygonal elements in $\mathcal{T}_{\mathtt{CFE}}$ is the same as the polynomial degree of the elements of $\mathcal{T}_h$. In the case of variable polynomial degrees, it is necessary to assume that the polynomial degree of each polygonal element is the same as the polynomial degree of all the children elements.

For each polygonal element $\kappa \in \mathcal{T}_{\mathtt{CFE}}$ we define $\hat{\kappa}$ as the smallest rectangle containing κ with edges parallel to the axes. Then the polynomial space $\mathcal{P}_p(\kappa)$ is defined to contain the polynomial functions in $\mathcal{P}_p(\hat{\kappa})$ restricted to the support of the element κ. So, the DG finite element space on the mesh $\mathcal{T}_{\mathtt{CFE}}$ is constructed gluing together the polynomial spaces $\mathcal{P}_p(\kappa)$ for all elements $\kappa \in \mathcal{T}_{\mathtt{CFE}}$, i.e.;

$$V(\mathcal{T}_{\mathtt{CFE}}, p) = \left\{u \in L_2(\Omega) : u|_\kappa \in \mathcal{P}_p(\kappa), \forall \kappa \in \mathcal{T}_{\mathtt{CFE}}\right\}.$$

In view of the definition of $V(\mathcal{T}_{\mathtt{CFE}}, p)$, it is clear that the DG space $V(\mathcal{T}_h, p)$ on the finer mesh simplify the construction of the finite element functions on polygons. Any polynomial function in $\mathcal{P}_p(\kappa)$ for any elements $\kappa \in \mathcal{T}_{\mathtt{CFE}}$ can be defined as a linear combination of the basis functions living on the children of κ, see [14, Section 8]. So, any integral on polygonal elements or on edges of polygonal elements can be computed as an integral on either elements or edges of the mesh $\mathcal{T}_h$. This is how the method assembles the discrete problem on polygonal meshes.

3. Composite discontinuous Galerkin finite element method

In this section, we introduce the *hp*-version of the (symmetric) interior penalty DGCFEM for the numerical approximation of (1). To this end, we first introduce the following notation.

We denote by $\mathcal{F}^{\mathcal{I}}_{\mathtt{CFE}}$ the set of all interior edges of the partition $\mathcal{T}_{\mathtt{CFE}}$ of Ω, and by $\mathcal{F}^{\mathcal{B}}_{\mathtt{CFE}}$ the set of all boundary edges of $\mathcal{T}_{\mathtt{CFE}}$. Furthermore, we define $\mathcal{F} = \mathcal{F}^{\mathcal{I}}_{\mathtt{CFE}} \cup \mathcal{F}^{\mathcal{B}}_{\mathtt{CFE}}$. The boundary $\partial\kappa$ of an element κ and the sets $\partial\kappa \setminus \partial\Omega$ and $\partial\kappa \cap \partial\Omega$ will be identified in a natural way with the corresponding subsets of $\mathcal{F}$. Let κ^+ and κ^- be two adjacent elements of $\mathcal{T}_{\mathtt{CFE}}$, and $\mathbf{x}$ an arbitrary point on the interior edge $F \in \mathcal{F}^{\mathcal{I}}_{\mathtt{CFE}}$ given by $F = \partial\kappa^+ \cap \partial\kappa^-$. Furthermore, let v and $\mathbf{q}$ be scalar- and vector-valued functions, respectively, that are smooth inside each element $\kappa^\pm$. By $(v^\pm, \mathbf{q}^\pm)$, we denote the traces of $(v, \mathbf{q})$ on F taken from within the interior of $\kappa^\pm$, respectively. Then, the averages of v and $\mathbf{q}$ at $\mathbf{x} \in F$ are given by

$$\{\{v\}\} = \frac{1}{2}(v^+ + v^-), \qquad \{\{\mathbf{q}\}\} = \frac{1}{2}(\mathbf{q}^+ + \mathbf{q}^-),$$

respectively. Similarly, the jumps of v and $\mathbf{q}$ at $\mathbf{x} \in F$ are given by

$$[\![v]\!] = v^+ \mathbf{n}_{\kappa^+} + v^- \mathbf{n}_{\kappa^-}, \qquad [\![\mathbf{q}]\!] = \mathbf{q}^+ \cdot \mathbf{n}_{\kappa^+} + \mathbf{q}^- \cdot \mathbf{n}_{\kappa^-},$$