Applied Mathematics and Computation 267 (2015) 664–681

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Approximate methods for stochastic eigenvalue problems

H. Hakula, V. Kaarnioja [1], M. Laaksonen *

Aalto University, School of Science, Department of Mathematics and Systems Analysis, P.O. Box 11100, FI-00076 Aalto, Finland

ARTICLE INFO

Keywords:
sFEM
Stochastic eigenvalue problem
Stochastic collocation
Stochastic Galerkin method
Inverse iteration
Sparse grids

ABSTRACT

We consider the discretization and solution of eigenvalue problems of elliptic operators with random coefficients. For solving the resulting systems of equations we present a new and efficient spectral inverse iteration based on the stochastic Galerkin approach with respect to a polynomial chaos basis. The curse of dimensionality inherent in normalization over parameter spaces is avoided by a solution of a non-linear system of equations defining the Galerkin coefficients. For reference we also present an algorithm for adaptive stochastic collocation. Functionality of the algorithms is demonstrated by applying them on four examples of a given model problem. Convergence of the Galerkin-based method is analyzed and the results are tested against the collocated reference solutions and theoretical predictions.

1. Introduction

Over the last decade numerical solution of stochastic partial differential equations has become a well-established field. The solution methods can broadly speaking be divided into *intrusive* and *non-intrusive* ones. The same division applies to stochastic eigenvalue problems as well. In this paper we focus on stochastic Galerkin method (intrusive) for a multiparametric problem. The benchmark method is a stochastic collocation algorithm (non-intrusive) by Andreev [1], a variant of which is also discussed here.

Stochastic eigenvalue problems (SEVP) arise in many applications. Ultimately our interests are in engineering applications, in particular effects of material models and manufacturing imperfections of geometric nature. SEVPs have attracted a lot of attention recently and various algorithms have been suggested for computing approximate eigenpairs: [2–4], especially the power iteration earlier in 2014 [5].

It should be noted that in the context of this paper it is assumed that the problems are positive definite and the eigenpair of interest is the ground state, that is, the one with the smallest simple eigenvalue. For linear elasticity the latter assumption does not hold and the algorithms must be extended to invariant subspaces in the future.

In stochastic eigenvalue problems one must address two central issues that do not arise in stochastic source problems: first, the eigenmodes are defined only up to a sign, and second, the eigenmodes must be normalized over the whole parameter space, that is, every realization must be normalized in the same way. The first issue is dealt with here by choosing a model problem where it is guaranteed that the eigenmode of interest does not change sign over the computational domain thus making global orientation over all realizations possible. The second issue is more problematic. The immediate solution

* Corresponding author.
E-mail addresses: harri.hakula@aalto.fi (H. Hakula), vesa.kaarnioja@aalto.fi (V. Kaarnioja), mikael.j.laaksonen@aalto.fi (M. Laaksonen).

[1] We acknowledge CSC IT Center for Science Ltd. for the allocation of computational resources (project ay6302). The author has been supported by the Academy of Finland (decision 267789).

http://dx.doi.org/10.1016/j.amc.2014.12.112
0096-3003/

is to ensure normalization over a quadrature, but regrettably this leads to the curse of dimensionality raising its ugly head and thus blurring the lines between collocation and Galerkin approaches. The main result of this paper is a new spectral inverse iteration based on efficient, though approximate, normalization procedure which avoids the curse of dimensionality by avoiding numerical integration via a solution of a non-linear system of equations defining the Galerkin coefficients. The effectiveness of this algorithm is demonstrated in the numerical experiments. The observed convergence rates in H_0^1 seminorm and L^2 norm are in line with theoretical predictions. It should be noted that the running times of the experiments on standard hardware are measured in minutes for Galerkin in contrast to hours or even days for collocation. The results indicate that future work in Galerkin schemes is warranted.

The rest of the paper is organized as follows: First the abstract problem is introduced in Section 2; The representation of the random input is given in Section 3; Section 4 is central to our discussion, both solution methods, collocation and spectral inverse iteration, are presented; Numerical experiments are discussed in Section 5 and finally conclusions are drawn in Section 6. Some useful definitions are given in the Appendix as well as the proof for the convergence of the spectral inverse iteration.

2. Problem formulation

The algorithms presented in this paper are designed for the following generalized stochastic matrix eigenvalue problem:

$$\mathbf{A}(\xi)\mathbf{u}(\xi) = \mu(\xi)\mathbf{B}\mathbf{u}(\xi), \tag{1}$$

with the eigenvector normalized as

$$\|\mathbf{u}(\xi)\|_{\mathbf{B}}^2 := \mathbf{u}(\xi)^{\mathrm{T}}\mathbf{B}\mathbf{u}(\xi) = 1. \tag{2}$$

Here, the stochastic dependence is represented by a vector of mutually independent random variables $\xi = (\xi_1, \xi_2, \ldots)$. The random matrix $\mathbf{A}(\xi)$ and the deterministic matrix $\mathbf{B}$ are assumed to be symmetric and positive definite. Furthermore, we will assume the dependence of the matrix $\mathbf{A}$ on the random vector ξ to take the following form

$$\mathbf{A}(\xi) = \mathbf{A}^{(0)} + \mathbf{A}^{(1)}\xi_1 + \mathbf{A}^{(2)}\xi_2 \ldots \tag{3}$$

with sufficient convergence of the series of deterministic matrices $\{\mathbf{A}^{(m)}\}_{m\geqslant 1}$. A similar expression with respect to an orthonormal polynomial chaos basis is also applicable.

Typically, we are interested in eigenvalue problems of infinite-dimensional operators with random coefficients. In this case, the stochastic matrix eigenvalue problem (1) should be considered as a finite-dimensional approximation of the original problem. Such an approximation can be acquired, for instance, with the help of the Karhunen–Loève expansion and a suitable spatial discretization. We will explain this procedure in more detail by means of a suitable model problem.

2.1. Model problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space, Ω being the set of outcomes, $\mathcal{F}$ a σ-algebra of events, and P a probability measure defined on Ω. We denote by $L_P^2(\Omega)$ the space of square integrable random variables on Ω. Furthermore, for a Hilbert space $\mathcal{H}$ we define the Bochner space

$$L_P^2(\Omega;\mathcal{H}) := \left\{ u : \Omega \to \mathcal{H} \;\middle|\; \int_\Omega \|u(\omega)\|_{\mathcal{H}}^2 dP(\omega) < \infty \right\}.$$

In the case that $\mathcal{H}$ is a separable Hilbert space, this admits the decomposition $L_P^2(\Omega;\mathcal{H}) \simeq L_P^2(\Omega) \otimes \mathcal{H}$.

Following the notational conventions from [1], we will take the eigenvalue problem of a diffusion operator with a random diffusion coefficient as our model problem. Let $D \subset \mathbb{R}^m$ be a bounded domain with a sufficiently smooth boundary and assume that the diffusion coefficient is a random field $a : \Omega \times D \to \mathbb{R}$. The diffusion coefficient is assumed to be strictly uniformly positive and uniformly bounded, i.e., for some positive constants $a_{\min}$ and $a_{\max}$ it holds that

$$P\left(\omega \in \Omega : a_{\min} \leqslant \operatorname*{essinf}_{x\in D} a(\omega, x) \leqslant \operatorname*{esssup}_{x\in D} a(\omega, x) \leqslant a_{\max}\right) = 1. \tag{4}$$

We now formulate the model problem as: find functions $\mu : \Omega \to \mathbb{R}$ and $u : \Omega \times D \to \mathbb{R}$ such that the equations

$$\begin{cases} -\nabla\cdot(a(\omega,\cdot)\nabla u(\omega,\cdot)) = \mu(\omega)u(\omega,\cdot) & \text{in } D \\ u(\omega,\cdot) = 0 & \text{on } \partial D \end{cases} \tag{5}$$

hold P-almost surely.

2.2. Spectra of deterministic realizations

For each fixed $\omega \in \Omega$ the eigenvalue problem (5) reduces to a single deterministic eigenvalue problem for which the variational form is given by: find $\mu(\omega) \in \mathbb{R}$ and $u(\omega,\cdot) \in H_0^1(D)$ such that