ARTICLE IN PRESS

Applied Mathematics and Computation xxx (2014) xxx–xxx

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the class of harmonic mappings which is related to the class of bounded boundary rotation

Yaşar Polatoğlu [a], Melike Aydoğan [b,*], Yasemin Kahramaner [c]

[a] Department of Mathematics and Computer Sciences, İstanbul Kültür University, İstanbul, Turkey
[b] Department of Mathematics, Işık University, Meşrutiyet Koyu, Şile, İstanbul, Turkey
[c] Department of Mathematics, İstanbul Ticaret University, İstanbul, Turkey

ARTICLE INFO

Keywords:
Functions with bounded boundary rotation
Distortion theorem
Growth theorem
Harmonic mapping

ABSTRACT

The class of bounded radius of rotation is generalization of the convex functions. The concept of functions bounded boundary rotation originated from Loewner (1917). But he did not use the present terminology. It was Paatero (1931, 1933) who systematically developed their properties and made an exhaustive study of the class V_k.

In the present paper we will investigate the class of harmonic mappings which is related to the class of bounded boundary rotation.

1. Introduction

Let Ω be the family of functions $\phi(z)$ regular and satisfying the conditions $\phi(0)=0$, $|\phi(z)|<1$ for all $z\in\mathbb{D}=\{z\in\mathbb{C}||z|<1\}$.

Next, denote by $\mathcal{P}$ the family of functions $p(z)=1+p_1z+p_2z^2++\ldots$ regular in $\mathbb{D}$ and such that $p(z)$ is in $\mathcal{P}$ if and only if

$$p(z)=\frac{1+\phi(z)}{1-\phi(z)} \tag{1.1}$$

for some function $\phi(z)\in\Omega$ and every $z\in\mathbb{D}$, and let P_k be the class of functions $p(z)$ defined in $\mathbb{D}$ satisfying the properties $p(0)=1$ and

$$\int_0^{2\pi}|Rep(z)|d\theta\leqslant k\pi \tag{1.2}$$

where $z=re^{i\theta},\ k\geqslant 2$, we can write (1.2) as

$$p(z)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t) \tag{1.3}$$

where $\mu(t)$ is a function with bounded variation on $[0,2\pi]$ such that

$$\int_0^{2\pi}d\mu(t)=2,\quad \int_0^{2\pi}|d\mu(t)|\leqslant k \tag{1.4}$$

* Corresponding author.
E-mail addresses: y.polatoglu@iku.edu.tr (Y. Polatoğlu), melike.aydogan@isikun.edu.tr (M. Aydoğan), ykahramaner@iticu.edu.tr (Y. Kahramaner).

http://dx.doi.org/10.1016/j.amc.2014.10.055
0096-3003/

From (1.2), we can write for $p(z) \in P_k$,

$$P(z) = \left(\frac{1}{2}+\frac{k}{4}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) \tag{1.5}$$

where $z \in \mathbb{D}$ and $p_1(z),\ p_2(z) \in P$ [1,5–8].

Moreover, let A denote the class of functions $s(z) = z + a_2z^2 + \ldots$ which are analytic in $\mathbb{D}$. Let $K,\ S^*$ and C denote the subclasses of A which are close-to-convex, starlike and convex in $\mathbb{D}$, respectively, and we define the following classes;

$$R_k = \left\{ s(z) | s(z) \in A, z\frac{s'(z)}{s(z)} \in P_k \right\} \tag{1.6}$$

$$V_k = \left\{ s(z) | s(z) \in A, \frac{(zs'(z))'}{s'(z)} \in P_k \right\} \tag{1.7}$$

We note that

$$s(z) \in V_k \Longleftrightarrow zs'(z) \in R_k \tag{1.8}$$

$$s(z) \in V_k \Longleftrightarrow 1 + z\frac{s''(z)}{s'(z)} = \left(\frac{1}{2}+\frac{k}{4}\right)p_1(z) + \left(\frac{1}{2}-\frac{k}{4}\right)p_2(z) \tag{1.9}$$

where $p_1(z),\ p_2(z) \in P$. Therefore we say that $s(z)$ belongs to V_k if and only if it is solution to the differential equation

$$1 + z\frac{s''(z)}{s'(z)} = p(z) \tag{1.10}$$

with initial data $s(0) = 0,\ s'(0) = 1$, where the function $p(z)$ is holomorphic in $\mathbb{D}$ such that $p(0) = 1$ and

$$lim_{r\to 1}\int_0^{2\pi} |Rep(re^{i\theta})| d\theta \leqslant k\pi \tag{1.11}$$

The classes V_k and R_k are called analytic functions with bounded boundary rotation and with bounded radius rotation, respectively. Also we note that the class V_k generalizes of convex functions and the class R_k generalizes of starlike functions [4].

Let $s_1(z)$ and $s_2(z)$ be elements of A. If there exists $\phi(z) \in \Omega$ such that $s_1(z) = s_2(\phi(z))$ for all $z \in \mathbb{D}$, then we say that $s_1(z)$ is subordinate to $s_2(z)$ and we write $s_1(z) \prec s_2(z)$. If $s_2(z)$ is univalent in $\mathbb{D}$, then $s_1(z) \prec s_2(z)$ if and only if $s_1(\mathbb{D}) \subset s_2(\mathbb{D}),\ s_1(0) = s_2(0)$ implies $s_1(\mathbb{D}_r) \prec s_2(\mathbb{D}_r)$, where $D_r = \{z | |z| < r,\ 0 < r < 1\}$. (Subordination and Lindelof Principle [3]).

Finally, a planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex valued harmonic function f, which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is simply connected domain the mapping f has canonical decomposition $f(z) = h(z) + \overline{g(z)}$ where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series

$$h(z) = \sum_{n=0}^{\infty} a_nz^n, g(z) = \sum_{n=0}^{\infty} b_nz^n, z \in \mathbb{D}.$$

where $a_n,\ b_n \in \mathbb{C},\ n = 0, 1, 2, 3, \ldots$. As in usual we call $h(z)$ is analytic part of f and $g(z)$ is co-analytic part of $f(z)$. An elegant and complete treatment the theory of harmonic mappings is given in Duren's monograph [2]. Lewy [2] proved in 1936 that the harmonic mapping $f(z)$ is locally univalent in $\mathbb{D}$ if and only if its jacobian $J_f = |h'(z)|^2 - |g'(z)|^2$ is different from zero in $\mathbb{D}$. In view of this result, locally univalent harmonic mappings in the open unit disc are either sense-reversing if $|g'(z)| > |h'(z)|$ in $\mathbb{D}$ or sense-preserving if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$. Throughout this paper we will restrict ourselves to the study of sense-preserving harmonic mappings. We also note that $f = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$ if and only if $h'(z)$ does not vanish in the unit disc $\mathbb{D}$, and the second complex dilatation $w(z) = \left(\frac{g'(z)}{h'(z)}\right)$ has the property $|w(z)| < 1$ in $\mathbb{D}$.

The class of all sense-preserving harmonic mappings of the open unit disc $\mathbb{D}$ with $a_0 = b_0 = 0$ and $a_1 = 1$ and will be denoted by S_H. Thus S_H contains the standard class S of analytic univalent functions. The family of all mappings $f \in S_H$ with the additional property that $g'(0) = 0$, i.e., $b_1 = 0$ is denoted by S_H^0. Thus it is clear that $S \subset S_H^0 \subset S_H$.

The main purpose of this paper is to investigate the following class

$$S_{HV(k,m)} = \left\{ f(z) = h(z) + \overline{g(z)} \in S_H | w(z) = \frac{g'(z)}{h'(z)} \prec b_1\left(\frac{1+z}{1-z}\right)^m, h(z) \in V_k \right\} \tag{1.12}$$

Therefore we will need the following lemma and theorem.

Lemma 1.1 [5]. *Let $\phi(z)$ be regular in the open unit disc $\mathbb{D}$, with $\phi(0) = 0$ Then if $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0, one has $z_0.\phi'(z_0) = k\phi(z_0)$ for some $k \geqslant 1$.*