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Lattice fractional calculus

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ABSTRACT

Integration and differentiation of non-integer orders for N-dimensional physical lattices with long-range particle interactions are suggested. The proposed lattice fractional derivatives and integrals are represented by kernels of lattice long-range interactions, such that their Fourier series transformations have a power-law form with respect to components of wave vector. Continuous limits for these lattice fractional derivatives and integrals give the continuum derivatives and integrals of non-integer orders with respect to coordinates. Lattice analogs of fractional differential equations that include suggested lattice differential and integral operators can serve as an important element of microscopic approach to nonlocal continuum models in mechanics and physics.

1. Introduction

The main approaches to describe nonlocal properties of media and materials are a macroscopic approach based on the continuum mechanics [1–5], and a microscopic approach based on the lattice mechanics [6–9]. Continuum mechanics can be considered as a continuous limit of lattice dynamics, where the sizes of continuum elements are much larger than the distances between lattice particles.

Theory of derivatives and integrals of non-integer orders [10–19] has a long history and it goes back to the famous scientist such as Leibniz, Riemann, Liouville, Letnikov, Weyl, Riesz and other. Fractional calculus and fractional differential equations have a wide application in different areas of physics [20–31]. Fractional integro-differential equations are very important to describe processes in nonlocal continua and media. Fractional integrals and differential operators with respect to coordinates allow us to describe continuously distributed system with power-law type of nonlocality. Therefore fractional calculus serve as a powerful tool in physics and mechanics of nonlocal continua. As it was shown in [41,42,25], the fractional differential equations for nonlocal continua can be directly connected to models of lattice with long-range interactions of power-law type. Interconnection between the equations for lattice with long-range interactions and the fractional differential equations for continuum is proved by special transform operator that includes a continuous limit, and the Fourier series and integral transformations [41–44]. In [55–59] this approach has been applied to lattice models of fractional nonlocal continua in one-dimensional case only. In this paper we propose a lattice fractional calculus that allows us to extend these lattice models to N-dimensional case.

Dynamics of physical lattices and discretely distributed systems with long-range interactions has been the subject of investigations in different areas of science. Effect of synchronization for nonlinear systems with long-range interactions is described in [32]. Non-equilibrium phase transitions for systems with long-range interactions are considered in [33]. Stationary states for fractional systems with long-range interactions are discussed in [46,34,35]. The evolution of soliton-like

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and breather-like structures in one-dimensional lattice of coupled oscillators with the long-range power are considered in [36]. Kinks in the Frenkel–Kontorova model with long-range particle interactions is studied in [37]. In statistical mechanics and nonlinear dynamics, solvable models with long-range interactions are described in detail in the reviews [38–40]. Different discrete systems and lattice with long-range interactions and its continuous limits are considered in [23,25]. It is important that lattice models with long-range interactions of power-law type can lead to fractional nonlocal continuum models in the continuous limit [41,42,25], Nonlocal continuum mechanics can be considered as a continuous limit of mechanics of lattice with long-range interactions, when the sizes of continuum element are much larger than the distances between particles of lattice.

It should be note that a calculus of operators of integer orders for physical lattice models has been considered in the papers [48–50]. This lattice calculus of integer order is defined on a general triangulating graph by using discrete field quantities and differential operators roughly analogous to differential forms and exterior differential calculus. A scheme to derive lattice differential operators of integer orders from the discrete velocities and associated Maxwell–Boltzmann distributions that are used in lattice hydrodynamics has been suggested in the articles [51,52]. In this paper to formulate a lattice fractional calculus, we use other approach that is based on models of physical lattices with long-range inter-particle interactions and its continuum limit that are suggested in [41,42,25] (see also [43–47,55–59]).

In this paper, we propose lattice analogs of differentiation and integration of non-integer orders based on N-dimensional generalization of the lattice approach suggested in [41,42,25]. A general form of lattice fractional derivatives and integrals that gives continuum derivatives and integrals of non-integer orders in continuous limit is suggested. These continuum fractional operators of differentiations and integrations can be considered as fractional derivatives and integrals of the Riesz type with respect to coordinates.

2. Lattice fractional differential operators

2.1. Lattice fractional partial derivatives

Let us consider an unbounded physical lattice characterized by N non-coplanar vectors $\mathbf{a}_i, i = 1, \dots, N$, that are the shortest vectors by which a lattice can be displaced and be brought back into itself. For simplification, we assume that $\mathbf{a}_i, i = 1, \dots, N$, are mutually perpendicular primitive lattice vectors. We choose directions of the axes of the Cartesian coordinate system coincides with the vector $\mathbf{a}_i$. Then $\mathbf{a}_i = a_i \mathbf{e}_i$, where $a_i = |\mathbf{a}_i|$ and $\mathbf{e}_i, i = 1, \dots, N$, is the basis of the Cartesian coordinate system for $\mathbb{R}^N$. This simplification means that the lattice is a primitive N-dimensional orthorhombic Bravais lattice. The position vector of an arbitrary lattice site is written $\mathbf{r}(\mathbf{n}) = \sum_{i=1}^{N} n_i \mathbf{a}_i$, where n_i are integer. In a lattice the sites are numbered by $\mathbf{n}$, so that the vector $\mathbf{n} = (n_1, \dots, n_N)$ can be considered as a number vector of the corresponding lattice particle. We assume that the equilibrium positions of particles coincide with the lattice sites $\mathbf{r}(\mathbf{n})$. Coordinates $\mathbf{r}(\mathbf{n})$ of lattice sites differs from the coordinates of the corresponding particles, when particles are displaced relative to their equilibrium positions. To define the coordinates of a particle, we define displacement of $\mathbf{n}$-particle from its equilibrium position by the scalar field $u(\mathbf{n})$, or the vector field $\mathbf{u}(\mathbf{n}) = \sum_{i=1}^{N} u_i(\mathbf{n}) \mathbf{e}_i$, where the vectors $\mathbf{e}_i = \mathbf{a}_i / |\mathbf{a}_i|$ form the basis of the Cartesian coordinate system. The functions $u_i(\mathbf{n}) = u_i(n_1, \dots, n_N)$ are components of the displacement vector for lattice particle that is defined by $\mathbf{n} = (n_1, \dots, n_N)$. In many cases, we can assume that $u(\mathbf{n})$ belongs to the Hilbert space l_2 of square-summable sequences to apply the Fourier transformations. For simplification, we will consider differential and integral operators for the lattice functions $u = u(\mathbf{n}) = u(n_1, \dots, n_N)$. All transformations can be easily generalized to the case of the vector functions.

Let us give a definition of lattice partial derivative of arbitrary positive real order α in the direction $\mathbf{e}_i = \mathbf{a}_i / |\mathbf{a}_i|$ in the lattice.

Definition 1. A lattice fractional partial derivative is the operator $\mathbb{D}_L^{\pm} \begin{bmatrix} \alpha \\ i \end{bmatrix}$ such that

$$\mathbb{D}_L^{\pm} \begin{bmatrix} \alpha \\ i \end{bmatrix} u = \frac{1}{a_i^{\alpha}} \sum_{m_i=-\infty}^{+\infty} K_{\alpha}^{\pm}(n_i - m_i)\, u(\mathbf{m}), \quad (i = 1, \dots, N), \tag{1}$$

where $\alpha \in \mathbb{R}, \alpha > 0$, $\mathbf{m} \in \mathbb{Z}$, and the interaction kernels $K_{\alpha}^{\pm}(n - m)$ are defined by the equations

$$K_{\alpha}^{+}(n - m) = \frac{\pi^{\alpha}}{\alpha + 1}\, {}_1F_2\left(\frac{\alpha + 1}{2}; \frac{1}{2}, \frac{\alpha + 3}{2}; -\frac{\pi^2 (n - m)^2}{4} \right), \quad \alpha > 0, \tag{2}$$

$$K_{\alpha}^{-}(n - m) = -\frac{\pi^{\alpha+1} (n - m)}{\alpha + 2}\, {}_1F_2\left(\frac{\alpha + 2}{2}; \frac{3}{2}, \frac{\alpha + 4}{2}; -\frac{\pi^2 (n - m)^2}{4} \right), \quad \alpha > 0, \tag{3}$$

where ${}_1F_2$ is the Gauss hypergeometric function [63]. The parameter $\alpha > 0$ will be called the order of the lattice derivative (1).

Let us explain the reasons for definition the interaction kernels $K_{\alpha}^{\pm}(n - m)$ in the forms (2), (3), and describe some properties of these kernels.