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Generalized convolution

J.A. Tenreiro Machado

Institute of Engineering of Polytechnic of Porto, Dept. of Electrical Engineering, Porto, Portugal

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ABSTRACT

The paper revisits the convolution operator and addresses its generalization in the perspective of fractional calculus. Two examples demonstrate the feasibility of the concept using analytical expressions and the inverse Fourier transform, for real and complex orders. Two approximate calculation schemes in the time domain are also tested.

1. Introduction

Fractional calculus (FC) is the scientific area that generalizes the standard integral and derivative operators up to real and complex orders. The historical origin of FC goes back to 1695, when L'Hospital asked about the meaning of $\mathcal{D}^{\frac{1}{2}}y$ and Leibniz replied with some comments about the apparent paradox. Many important mathematician developed the topic during three centuries [1–6], but only in the last decades applied sciences recognized the importance of the tool to model dynamical phenomena with long range memory effects [7–13]. Significant progresses took place in numerical analysis and signal processing [14–19], but many important aspects still remain to be explored. For example, the interpretation of the fractional derivative or integral is still the object of strong debate and several perspectives were formulated [20–32].

The concepts behind the generalization of the concept of derivatives and integrals can be applied with other operators. Several studies addressed the fractional convolution in the scope of the fractional Fourier transform [33–40]. However, the study of convolution in signal analysis having in mind the FC tools has not received significant attention. This paper addresses the topic of generalizing that operator and evaluating its practical implementation in signal processing.

Bearing these ideas in mind, the paper is organized as follows. Section 2 discusses the convolution in the perspective of FC. Several examples are studied and two approximate techniques for its calculation are proposed. Finally, Section 3 outlines the main conclusions and points towards future work.

2. Convolution in the perspective of fractional calculus

The convolution of two signals $f(t)$ and $g(t)$ is defined as:

$$\mathcal{C}(f,g)(t) = (f * g)(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{+\infty} f(t-\tau)g(\tau)d\tau, \tag{1}$$

where t denotes time, $*$ is the standard symbol for the operator and $\mathcal{C}(\cdot)$ is the notation adopted in the sequel.

When f and g are defined for $t \geqslant 0$ expression (1) reduces to:

$$\mathcal{C}(f,g)(t) = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau. \tag{2}$$

E-mail address: jtm@isep.ipp.pt

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Convolution is usually interpreted as the overlapping area between the two functions when one of them is flipped and shifted by t.

The product of the Laplace transforms of the signals, $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, is related with convolution by the property:

$$\mathcal{C}(f,g)(t) = \mathcal{L}^{-1}\{F(s)\cdot G(s)\}, \tag{3}$$

where s denotes the Laplace variable.

When $f = g$ expression (3) reduces to:

$$\mathcal{C}_*(f)(t) = \mathcal{C}(f,f)(t) = \mathcal{L}^{-1}\{F^2(s)\}. \tag{4}$$

Repeating the procedure leads to the expression:

$$\mathcal{C}_*^n(f)(t) = \mathcal{C}(f,\mathcal{C}_*^{n-1}(f))(t) = \mathcal{L}^{-1}\{F^{2n}(s)\}, \tag{5}$$

where $\mathcal{C}_*^n(f)(t)$ represents the $n \in \mathbb{N}$ order convolution of f with itself.

Within the scope of FC Eq. (5) can be extended to order $\gamma \in \mathbb{C}$:

$$\mathcal{C}^\gamma(f,g)(t) = \mathcal{L}^{-1}\{[F(s)\cdot G(s)]^\gamma\} \tag{6}$$

and

$$\mathcal{C}_*^\gamma(f)(t) = \mathcal{L}^{-1}\{F^{2\gamma}(s)\} \tag{7}$$

We shall denote the operation by *generalized convolution*.

Expression (6) means that if $F(s) = \frac{1}{s}$ and $G(s) = H^{\frac{1}{\gamma}}(s)$, then we have:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^\gamma}\cdot H(s)\right\} = \mathcal{C}^\gamma(u_\gamma, h)(t) = \mathcal{D}^{-\gamma}\{h(t)\}, \tag{8}$$

where $u_\gamma(t) = \mathcal{L}^{-1}\{\frac{1}{s^\gamma}\}$ and $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

In this line of thought, (8) establishes a relationship between fractional integral $\mathcal{D}^{-\gamma}$ and generalized convolution $\mathcal{C}^\gamma$. This result may shed some extra light into the interpretation of fractional derivatives [29–31]. Recently it was proposed the physical interpretation of a fractional trajectory as being an average over an ensemble of stochastic trajectories [41,42]. Therefore, the convolution operation can be adopted in the description of physical phenomena involving strong discontinuities, that occur in phenomena such as collisions, up to smooth evolutions, usual in systems with slow dynamics.

Let us consider the following example [43] with $Re(s) > \max(-a,-b)$ and $t \geqslant 0$:

$$G(s) = \frac{1}{s+a}\cdot\frac{1}{s+b} \tag{9}$$

$$\mathcal{L}^{-1}\{G^\alpha(s)\} = \mathcal{C}^\alpha(e^{-at}, e^{-bt}) = \frac{\sqrt{\pi}}{\Gamma(\alpha)}\left(\frac{t}{a-b}\right)^{\alpha-\frac{1}{2}} e^{-\frac{a+b}{2}t} I_{\alpha-\frac{1}{2}}\left(\frac{a-b}{2}t\right) \tag{10}$$

where $\alpha \in \mathbb{R}$, $\Gamma(\cdot)$ denotes the Gamma function and $I_\alpha(\cdot)$ is the modified Bessel function of the first kind.

For the integer orders $\alpha = 1$ and $\alpha = 2$ expression (10) simplifies to

$$\mathcal{L}^{-1}\{G(s)\} = \mathcal{C}(e^{-at}, e^{-bt}) = \frac{e^{-at} - e^{-bt}}{b-a} \tag{11}$$

$$\mathcal{L}^{-1}\{G^2(s)\} = \mathcal{C}^2(e^{-at}, e^{-bt}) = \frac{1}{(a-b)^2}\left[\left(t + \frac{2}{a-b}\right)e^{-at} + \left(t - \frac{2}{a-b}\right)e^{-bt}\right] \tag{12}$$

Fig. 1 shows the impulse response of expression (10) for $\alpha = \{\frac{1}{2}, 1, \frac{3}{2}, 2\}$.

According with (8) the expressions can be interpreted not only as standard inverse Laplace results, but also as fractional integral $\mathcal{D}^{-\gamma}\left(\frac{e^{-at}-e^{-bt}}{b-a}\right)$.

We observe a considerable variation of the charts for low values of α, particularly at the initial transient, in opposition with a more conservative behavior for larger values of α and for steady-state.

In the example expression (10) was used and, therefore, the generalized convolution was not calculated explicitly. Therefore, in spite of the straightforward generalization, it remains the problem of calculation for the cases where a closed form solution is not available.

Let us suppose a second example with the $sinc(\cdot)$ function:

$$\mathcal{F}\{\Pi_\alpha(t)\} = \left[\frac{\sin(\omega T)}{\omega T}\right]^\alpha = sinc^\alpha(\omega T) \tag{13}$$

where $\alpha \in \mathbb{R}$, and ω and $\mathcal{F}\{\cdot\}$ denote the Fourier variable and operator, respectively.

For $\alpha = \{0, 1, 2, 3\}$ we obtain the Dirac, rectangular, triangular and parabolic functions: