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# Stability and convergence of a new finite volume method for a two-sided space-fractional diffusion equation

L.B. Feng [a], P. Zhuang [a,c], F. Liu [b,*], I. Turner [b]

[a] School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
[b] School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, Qld. 4001, Australia
[c] Fujian Provincial Key Laboratory of Mathematical Modeling and High-Performance Scientific Computation, Xiamen University, Xiamen 361005, China

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ABSTRACT

In this paper, we consider a two-sided space-fractional diffusion equation with variable coefficients on a finite domain. Firstly, based on the nodal basis functions, we present a new fractional finite volume method for the two-sided space-fractional diffusion equation and derive the implicit scheme and solve it in matrix form. Secondly, we prove the stability and convergence of the implicit fractional finite volume method and conclude that the method is unconditionally stable and convergent. Finally, some numerical examples are given to show the effectiveness of the new numerical method, and the results are in excellent agreement with theoretical analysis.



## 1. Introduction

There has been increasing interests in the description of physical and chemical process by means of equations involving fractional derivatives over the last decades. And, fractional derivatives have been successfully applied into many sciences, such as the fractional diffusion and wave equations [1–4], subdiffusion and superdiffusion equations [5,6], electrical systems [7], bioengineering [8], system biology [9], chemistry and biochemistry [10], hydrology [11–14], and finance [15–18]. In the area of physics, fractional space derivatives are utilized to model anomalous diffusion or dispersion, where a particle spreads at a rate inconsistent with the classical Brownian motion model [5]. Particularly, the Riesz fractional derivative includes a left-sided Riemann–Liouville derivative and a right-sided Riemann–Liouville derivative that allow the modeling of flow regime impacts from either side of the domain [19].

In this paper, we consider the following two-sided space-fractional diffusion equation with variable coefficients:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{\mathscr{C}(x)\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \mathscr{D}(x)\frac{\partial^\alpha u(x,t)}{\partial(-x)^\alpha}\right\} + f(x,t), \tag{1}$$

subject to the initial condition

$$u(x,0) = \psi(x), \quad 0 \leqslant x \leqslant L \tag{2}$$

and the zero Dirichlet boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{3}$$

* Corresponding author.
*E-mail addresses:* fenglibo2012@126.com (L.B. Feng), zxy1104@xmu.edu.cn (P. Zhuang), f.liu@qut.edu.au (F. Liu), i.turner@qut.edu.au (I. Turner).

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where $0 < \alpha < 1, \mathscr{C}(x)$ and $\mathscr{D}(x)$ are the nonnegative diffusion coefficients, and $f(x,t)$ is a source term. The operators $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha}, \frac{\partial^\alpha u(x,t)}{\partial(-x)^\alpha}$ are the left and right Riemann–Liouville fractional derivatives on a finite domain $[a,b]$ defined by

$$\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_a^x (x-\xi)^{-\alpha} u(\xi,t) d\xi,$$

$$\frac{\partial^\alpha u(x,t)}{\partial(-x)^\alpha} = \frac{-1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_x^b (\xi-x)^{-\alpha} u(\xi,t) d\xi,$$

where $\Gamma(\cdot)$ represents the Euler gamma function.

**Remark.** When the variable coefficients considered in Eq. (1) take the following special form

$$\mathscr{C}(x) = \mathscr{D}(x) = -\frac{k}{2\cos(\pi(1+\alpha)/2)}, \quad k > 0,$$

a space-fractional diffusion equation with Riesz fractional operator can be obtained as

$$\frac{\partial u(x,t)}{\partial t} = k \frac{\partial^{1+\alpha} u(x,t)}{\partial |x|^{1+\alpha}} + f(x,t), \tag{4}$$

where the Riesz fractional operator $\frac{\partial^{1+\alpha} u}{\partial |x|^{1+\alpha}}$ is defined as

$$\frac{\partial^{1+\alpha} u(x,t)}{\partial |x|^{1+\alpha}} = -\frac{1}{2\cos\frac{\pi(1+\alpha)}{2}} \left[ \frac{\partial^{1+\alpha} u(x,t)}{\partial x^{1+\alpha}} + \frac{\partial^{1+\alpha} u(x,t)}{\partial(-x)^{1+\alpha}} \right], \tag{5}$$

The Riesz fractional derivative is a symmetric fractional derivative, which was derived from the kinetics of chaotic dynamics by Saichev and Zaslavsky [20] and summarized by Zaslavsky [19]. Generally, the Grünwald–Letnikov derivative is used to approximate to the Riesz fractional derivative, most of which are finite difference methods [21–24,26,27,29,37,38]. In addition, the matrix transform method [25,35], Galerkin finite element method [28], predictor–corrector method [36], variational iteration method [39] and alternating direction method [30] are also proposed to applied to the fractional diffusion equations with Riesz space fractional derivative.

Applying the finite volume method to solve the fractional diffusion equations is sparse. Hejazi et al. [31] and Liu et al. [32] proposed the finite volume method for solving the fractional diffusion equations respectively, and Yang et al. [34] extend the finite volume method to the two dimensional fractional diffusion equation, all of which are without theoretical analysis. Recently, Hejazi et al. [33] employed the finite volume method for the space advection–dispersion equation. They utilized fractionally-shifted Grünwald formulae for the fractional derivative and proved the stability and convergence of the scheme, whose accuracy is $O(\tau + h)$. To the best of the authors' knowledge, stability and convergence of the finite volume method based on the nodal basis functions to the two-sided space-fractional diffusion equation are still not reported in the literature. Totally different from [33], in this paper, we propose a new finite volume method based on the nodal basis functions for the two-sided space-fractional diffusion equation, and obtain the implicit finite volume method. Furthermore, we prove that the new implicit finite volume method for the two-sided space-fractional diffusion equation is unconditionally stable and convergent with the accuracy of $O(\tau + h^2)$.

The outline of the paper is as follows. In Section 2, we first present a new fractional finite volume method for the two-sided space-fractional diffusion equation, and then the implicit fractional finite volume method based on the nodal basis functions is derived. We prove the stability and convergence of the implicit fractional finite volume method for Eq. (4) in Section 3. Finally, some numerical results for the finite volume method are carried out and the results are compared with the exact solution.

## 2. A new fractional finite volume method

In this section, we propose a new finite volume method using nodal basis functions for solving the two-sided space-fractional diffusion equation, which is subject to zero Dirichlet boundary conditions.

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x} \left\{ \mathscr{C}(x) \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \mathscr{D}(x) \frac{\partial^\alpha u(x,t)}{\partial(-x)^\alpha} \right\} + f(x,t), \tag{6}$$

We define $t_n = n\tau, n = 0, 1, \ldots, N$, let $\Omega = [0, L]$ be a finite domain, setting $S_h$ be a uniform partition of $\Omega$, which is given by $x_i = ih$ for $i = 0, 1, \ldots, m$, where $\tau = T/N$ and $h = L/m$ are the time and space steps, respectively. First, we present the semi-discrete form of Eq. (6), the implicit Euler scheme:

$$\frac{u(x,t_n) - u(x,t_{n-1})}{\tau} = \frac{\partial}{\partial x} \left\{ \mathscr{C}(x) \frac{\partial^\alpha u(x,t_n)}{\partial x^\alpha} - \mathscr{D}(x) \frac{\partial^\alpha u(x,t_n)}{\partial(-x)^\alpha} \right\} + f(x,t_n). \tag{7}$$