Computing Hadamard type operators of variable fractional order

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ARTICLE INFO	ABSTRACT
	We consider Hadamard fractional derivatives and integrals of variable fractional order. A new type of fractional operator, which we call the Hadamard–Marchaud fractional derivative, is also considered. The objective is to represent these operators as series of terms involving integer-order derivatives only, and then approximate the fractional operators by a finite sum. An upper bound formula for the error is provided. We exemplify our method by applying the proposed numerical procedure to the solution of a fractional differential equation and a fractional variational problem with dependence on the Hadamard–Marchaud fractional derivative. © 2014 Elsevier Inc. All rights reserved.

1. Introduction

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number powers or complex number powers of the differentiation and integration operators [26,29]. It has been called "The calculus of the XXI century" (K. Nishimoto, 1989) and claimed that "Nature works with fractional time derivatives" (S. Westerlund, 1991) [28]. Several definitions for fractional derivatives and fractional integrals are found in the literature. Although the most common ones seem to be the Riemann–Liouville and Caputo fractional operators, recently there has been an increasing interest in the development of Hadamard's XIX century fractional calculus [11]: see [1,4–7,12–16] and references therein. This calculus is due to the French mathematician Jacques Hadamard (1865–1963), where instead of power functions, as in Riemann–Liouville and Caputo fractional calculi, one has logarithm functions. The left and right Hadamard fractional integrals of order $\alpha > 0$ are defined by

$$
{}_a\mathcal{I}_t^{\alpha(t)}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} \frac{x(\tau)}{\tau}\, d\tau,
$$

and

$$
{}_t\mathcal{I}_b^{\alpha(t)}x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\ln\frac{\tau}{t}\right)^{\alpha-1} \frac{x(\tau)}{\tau}\, d\tau,
$$

respectively, while the left and right Hadamard fractional derivatives of order $\alpha \in (0,1)$ are given by

$$
{}_t\mathcal{D}_b^{\alpha(t)}x(t) = \frac{t}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \left(\ln\frac{t}{\tau}\right)^{-\alpha} \frac{x(\tau)}{\tau}\, d\tau, \tag{1}
$$

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and

$$_a\mathcal{D}_t^{\alpha(t)}x(t) = \frac{-t}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b\left(\ln\frac{\tau}{t}\right)^{-\alpha}\frac{x(\tau)}{\tau}d\tau,$$

respectively. Uniqueness and continuous dependence of solutions for nonlinear fractional differential systems with Hadamard derivatives is discussed in [18]. The main purpose of this paper is to extend the previous definitions to the case where the order α of the integrals and of the derivatives is not a constant, but a function that depends on time. Such time-dependence of α has already been considered for Riemann–Liouville and Caputo fractional operators, and has proven to describe better certain phenomena (see, e.g., [8–10,17,21,24,25,27]). To the best of our knowledge, an extension to Hadamard fractional operators is new and no work has been carried out so far in this direction. This is due to practical difficulties in computing such fractional derivatives and integrals of variable order. For this reason, here we propose a simple but effective numerical method that allows to deal with variable fractional order operators of Hadamard type.

The organization of the paper is the following. In Section 2 we extend known definitions of Hadamard fractional operators by considering the order α to be a function, and present a new definition of derivative, the Hadamard–Marchaud fractional derivative, which is an intrinsic variable order operator. In Section 3 we prove expansion formulas for the given fractional operators, using only integer-order derivatives. Finally, in Section 4 we give some concrete examples of the usefulness of the proposed method, including the application to the solution of a fractional differential system of variable order (Section 4.1) and to the solution of a fractional variational problem of variable order (Section 4.2).

2. Hadamard operators of variable fractional order

Along the text, the order of fractional operators is given by a function $\alpha \in C^1([a,b],(0,1))$, and the space of functions $x : [a,b] \to \mathbb{R}$ is such that each of the following integrals are well-defined, where a,b are two reals with $0 < a < b$.

Definition 2.1 (*Hadamard integrals of variable fractional order*). The left and right Hadamard fractional integrals of order $\alpha(t)$ are defined by

$$_a\mathcal{I}_t^{\alpha(t)}x(t) = \frac{1}{\Gamma(\alpha(t))}\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha(t)-1}\frac{x(\tau)}{\tau}d\tau,$$

and

$$_t\mathcal{I}_b^{\alpha(t)}x(t) = \frac{1}{\Gamma(\alpha(t))}\int_t^b\left(\ln\frac{\tau}{t}\right)^{\alpha(t)-1}\frac{x(\tau)}{\tau}d\tau,$$

respectively.

Definition 2.2 (*Hadamard derivatives of variable fractional order*). The left and right Hadamard fractional derivatives of order $\alpha(t)$ are defined by

$$_a\mathcal{D}_t^{\alpha(t)}x(t) = \frac{t}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_a^t\left(\ln\frac{t}{\tau}\right)^{-\alpha(t)}\frac{x(\tau)}{\tau}d\tau,$$

and

$$_t\mathcal{D}_b^{\alpha(t)}x(t) = \frac{-t}{\Gamma(1-\alpha(t))}\frac{d}{dt}\int_t^b\left(\ln\frac{\tau}{t}\right)^{-\alpha(t)}\frac{x(\tau)}{\tau}d\tau,$$

respectively.

The two Definitions 2.1 and 2.2 coincide with the classical definitions of Hadamard when the order $\alpha(t)$ is a constant function. Besides these definitions, we introduce a different one inspired on Hadamard and Marchaud fractional derivatives [26].

Definition 2.3 (*Hadamard–Marchaud derivatives of variable fractional order*). The left and right Hadamard–Marchaud fractional derivatives of order $\alpha(t)$ are defined by

$$_a\mathbb{D}_t^{\alpha(t)}x(t) = \frac{x(t)}{\Gamma(1-\alpha(t))}\left(\ln\frac{t}{a}\right)^{-\alpha(t)} + \frac{\alpha(t)}{\Gamma(1-\alpha(t))}\int_a^t\frac{x(t)-x(\tau)}{\tau}\left(\ln\frac{t}{\tau}\right)^{-\alpha(t)-1}d\tau, \tag{2}$$

and

$$_t\mathbb{D}_b^{\alpha(t)}x(t) = \frac{x(t)}{\Gamma(1-\alpha(t))}\left(\ln\frac{b}{t}\right)^{-\alpha(t)} + \frac{\alpha(t)}{\Gamma(1-\alpha(t))}\int_t^b\frac{x(t)-x(\tau)}{\tau}\left(\ln\frac{\tau}{t}\right)^{-\alpha(t)-1}d\tau, \tag{3}$$

respectively.