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Suppressing chaos in discontinuous systems of fractional order by active control

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ABSTRACT

In this paper, a chaos control algorithm for a class of piece-wise continuous chaotic systems of fractional order, in the Caputo sense, is proposed. With the aid of Filippov's convex regularization and via differential inclusions, the underlying discontinuous initial value problem is first recast in terms of a set-valued problem and hence it is continuously approximated by using Cellina's Theorem for differential inclusions. For chaos control, an active control technique is implemented so that the unstable equilibria become stable. As example, Shimizu–Morioka's system is considered. Numerical simulations are obtained by means of the Adams–Bashforth–Moulton method for differential equations of fractional-order.

1. Introduction

Several real-life systems show non-smooth physical properties (for instance dry friction, forced vibration brake processes with locking phase, stick, and slip phenomena) which can be suitably modeled by introducing some kind of discontinuity. Moreover, anomalous processes (for instance in non-standard materials) exhibit *memory* and *hereditary* properties and derivatives of fractional order are an effective tool to keep into account these phenomena.

Thus, fractional discontinuous systems provide a logical and attractive link between systems of fractional order and discontinuous systems.

Chaos control in continuous fractional-order systems, have been realized for many systems such as: Lorenz system, Chua system, Rossler system, Chen system, Liu system, Rabinovich–Fabrikant system, Coullet system, dynamos system, Duffing system, Arneodo system, Newton–Leipnik system and so on (few of the numerous related papers are [1–5]).

Anyway, because of the lack of numerical methods specifically devised for fractional differential equations (FDEs) with discontinuous right-hand side, discontinuous systems of fractional-order have not been rigorously studied.

In [6] it was investigated the behavior of some classical methods for discontinuous FDEs under a set-valued regularization into the Filippov's framework [7,8]; in particular, the chattering-free behavior of the generalization of the implicit Euler scheme was showed.

Solving problems resulting from the Filippov set-valued regularization is however not a simple task and requires, in most cases, to recast the original problem in a linear complementarity problem whose numerical solution is often very demanding.

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A possible way to remove these obstacles, is to approximate continuously the underlying initial value problem by modeling the discontinuous system according to the algorithm presented in [9]. In this way the chaos control problem becomes a standard chaos control of continuous systems of fractional-order.

In this paper we focus on discontinuous problems in which the right-hand side is a piece-wise continuous (PWC) function $f:\mathbb{R}^n\to\mathbb{R}^n$ having the following form

$$f(x(t)) = g(x(t)) + Kx(t) + A(x(t))s(x(t)), \tag{1}$$

where $g:\mathbb{R}^n\to\mathbb{R}^n$ is a nonlinear and (at least) continuous function, $s:\mathbb{R}^n\to\mathbb{R}^n$ a piece-wise continuous function $s(x)=(s_1(x_1),s_2(x_2),\dots,s_n(x_n))^T$ with $s_i:\mathbb{R}\to\mathbb{R}, i=1,2,\dots,n$ real piece-wise constant functions, $A\in\mathbb{R}^{n\times n}$ a square matrix of real continuous functions and K a square real matrix, representing the linear part of f.

Discontinuous systems of fractional-order are modeled in this paper by the following initial value problem (IVP)

$$D_*^q x(t) = f(x(t)), \quad x(0)=x_0, \quad t\in I=[0,\infty), \tag{2}$$

where f is the PWC function defined by (1) and, for $q=(q_1,q_2,\dots,q_n), D_*^q x(t) = (D_*^{q_1}x_1(t), D_*^{q_2}x_2(t),\dots,D_*^{q_n}x_n(t))$ denotes the vector of the differential operator of fractional order q_i applied to each component of $x(t)$.

In the past several alternative definitions have been proposed to provide valuable generalizations of differential operators to non integer order. Although the approach named as Riemann–Liouville is the most important both for theoretical and historical reasons, for practical applications the definition due to Caputo [10] is the most appropriate and useful. Indeed, the Caputo's fractional derivative has the considerable advantage of allowing to couple differential equations with classical initial conditions of Cauchy type as in (2) which not only have a clearly interpretable physical meaning [11] but can also be measured to proper initializing the simulation.

Since chaotic fractional-order systems are usually modeled with subunit fractional orders $0<q_i\leqslant 1, i=1,2,\dots,n$, the Caputo's differential operator of order q_i, with respect to the starting point 0, is defined as [10,12,13]

$$D_*^{q_i}x_i(t) = \frac{1}{\Gamma(1-q_i)}\int_0^t (t-\tau)^{-q_i}\frac{d}{dt}x_i(\tau)d\tau,$$

where $\Gamma(z)$ is the Euler's Gamma function.

Regarding the matrix A, the following assumption is considered.

(**H1**) $A(x)s(x)$ is discontinuous in at least one of his components.

The discontinuity impediment can be avoided by using the Filippov's technique [7] to convert a single valued (discontinuous) problem into a set-valued one. Then, via Cellina's Theorem for differential inclusions [14,15], set-valued functions can be continuously approximated in small neighborhoods.

The continuous approximation algorithm proposed in this paper regards the discontinuous functions s_i being valid for a large class of functions such as the *Heaviside* function, the *rectangular* function (as difference of two Heaviside functions), or the *signum*, one of the most encountered PWC functions in practical applications.

Standard techniques can hence be applied to the continuous approximation to device an active control in order to suppress the appearance of chaos.

The paper is organized as follows: Section 2 deals with the approximation of the PWC function (1) and shows how the IVP (2) can be transformed into a continuous problem. Section 3 concerns the investigation of stability issues of the approximated continuous problem. In Section 4 the chaos control obtained by stabilizing unstable equilibria of PWC systems of fractional-order is investigated. The application of these techniques to the fractional-order variant of Shimizu–Morioka's system (4) is hence analyzed in Section 5.

2. Continuous approximation of PWC systems of fractional-order

Notation 1. Let $\mathcal{M}$ be the discontinuity set of f, generated by the discontinuity points of the components s_i.

Example 2.1. For the linear PWC function $f:\mathbb{R}\to\mathbb{R}$

$$f(x) = 2-3sgn(x), \tag{3}$$

the discontinuity set is $\mathcal{M}=\{0\}$ and determines on $\mathbb{R}$ the continuity sub-domains $\mathcal{D}_1=(-\infty,0]$ and $\mathcal{D}_2=[0,\infty)$ (see Fig. 1).

The example of PWC systems analyzed in this paper, is the fractional variant of the chaotic Shimizu–Morioka's three-dimensional system [16,17]

$$\begin{aligned} D_*^{q_1}x_1 &= x_2,\\ D_*^{q_2}x_2 &= (1-x_3)sgn(x_1) - ax_2,\\ D_*^{q_3}x_3 &= x_1^2 - bx_3, \end{aligned} \tag{4}$$

with $a=0.75$ and $b=0.45$. Here $g(x)=(0,0,x_1^2)$ and