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A novel expansion iterative method for solving linear partial differential equations of fractional order

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ABSTRACT

In this manuscript, we implement a relatively new analytic iterative technique for solving time–space-fractional linear partial differential equations subject to given constraints conditions based on the generalized Taylor series formula. The solution methodology is based on generating the multiple fractional power series expansion solution in the form of a rapidly convergent series with minimum size of calculations. This method can be used as an alternative to obtain analytic solutions of different types of fractional linear partial differential equations applied in mathematics, physics, and engineering. Some numerical test applications were analyzed to illustrate the procedure and to confirm the performance of the proposed method in order to show its potentiality, generality, and accuracy for solving such equations with different constraints conditions. Numerical results coupled with graphical representations explicitly reveal the complete reliability and efficiency of the suggested algorithm.

1. Introduction

Fractional partial differential equations (PDEs) are found to be an effective tool to describe certain physical phenomena such as, damping laws, rheology, diffusion processes, electrostatics, electrodynamics, fluid flow, elasticity, and so on [1–7]. Problems in fractional PDEs are not only important but also quite challenging which usually involves hard mathematical solution techniques. Anyhow, in most real-life applications, it is too complicated to obtain exact solutions to PDEs of fractional order in terms of composite elementary functions in a simple manner, so an efficient, reliable numerical algorithm for the solutions of such equations is required; it is little wonder that with the development of fast, efficient digital computers, the role of numerical methods in mathematics, physics, and engineering problem solving has increased dramatically in recent years. The objective of the present letter is to extend the application of the iterative residual power series (RPS) method [8–14] to provide analytic solutions for initial value problems of linear PDEs of fractional order and to make comparisons with some of the well-known analytic methods.

The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order, which unifies and generalizes the notions of integer-order differentiation and n-fold integration. Theory of fractional calculus is a significantly

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important and useful branch of mathematics having a broad range of applications at almost any branch of science. Techniques of fractional calculus have been employed at the modeling of many different phenomena in mathematics, physics, and engineering [15–21]. The most important advantage of using fractional calculus in these and other applications is their nonlocal property. It is well known that the integer order differential operators and the integer order integral operators are local, while on the other hand, the fractional order differential operators and the fractional order integral operators are nonlocal. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. In fact, this is main reason why differential operators of fractional order provide an excellent instrument for description of memory and hereditary properties of various mathematical, physical, and engineering processes.

Series expansions are very important aids in numerical calculations, especially for quick estimates made in hand calculation, for example, in evaluating functions, integrals, or derivatives. Solutions to fractional PDEs can often be expressed in terms of series expansions. Since, the advent of computers, it has, however, become more common to treat fractional PDEs directly, using different approximation method instead of series expansions. But in connection with the development of automatic methods for formula manipulation, one can anticipate renewed interest for series methods. These methods have some advantages, especially in multidimensional solutions for PDEs of fractional order.

The RPS method was developed as an efficient method for determine values of coefficients of the power series solution for the first-order and the second-order fuzzy DEs [8]. It has been successfully applied in the numerical solution of the generalized Lane-Emden equation, which is a highly nonlinear singular DE [9], in the numerical solution of higher-order regular DEs [10], in the solution of composite and noncomposite fractional DEs [11], in predict and represent the multiplicity of solutions to boundary value problems of fractional order [12], in construct and predict the solitary pattern solutions for nonlinear time-fractional dispersive PDEs [13], and in approximate solution of the nonlinear fractional KdV-Burgers equation [14]. The RPS method is effective and easy to construct power series solution for strongly linear and nonlinear equations without linearization, perturbation, or discretization. Different from the classical power series method, the RPS method does not need to compare the coefficients of the corresponding terms and a recursion relation is not required. This method computes the coefficients of the power series by a chain of algebraic equations of one or more variables. In fact, the RPS method is an alternative procedure for obtaining analytic solutions for PDEs of fractional order. By using residual error concept, we get a series solutions; in practice truncated series solutions. Moreover, the obtained solutions and all their time–space-fractional derivatives are applicable for each arbitrary point and each multidimensional variable in the given domain. On the other aspect as well, the RPS method does not require any converting while switching from the low-order to the higher-order; as a result the method can be applied directly to the given problem by choosing an appropriate initial guess approximation.

In the present paper, the RPS method will investigate to construct a new algorithm for finding analytical solutions to the following classes of higher-order linear fractional PDEs of the general form

$$D_t^{\alpha}u(x,t) = L[u(x,t)],\ x \in \mathbb{R},\ t \geqslant t_0,\ 0 \leqslant m-1 < \alpha \leqslant m,\ m \in \mathbb{N}, \tag{1.1}$$

subject to the nonhomogeneous initial conditions

$$\frac{\partial^j u(x,t_0)}{\partial t^j} = \varphi_j(x),\ x \in \mathbb{R},\quad j = 0,1,2,\ldots,m-1, \tag{1.2}$$

and subject to the constraint linear differential operator

$$L[u(x,t)] = \sum_{j=0}^{m-1}\left[g_j(x)t^j \frac{\partial^j}{\partial t^j}\left(\frac{\partial^{k_j} u(x,t)}{\partial x^{k_j}}\right)\right] + \sum_{i=0}^{\infty}\sum_{j=0}^{m-1}\frac{h_{ij}(x)}{\Gamma(i\alpha+j+1)}(t-t_0)^{j+i\alpha},\quad k_j \in \mathbb{N}, \tag{1.3}$$

where D_t^{α} is the Caputo fractional derivative of order α and $\varphi_j(x), g_j(x), h_{ij}(x)$ are given analytic functions on $\mathbb{R}$. Throughout this paper, $\mathbb{N}$ the set of integer numbers, $\mathbb{R}$ the set of real numbers, and Γ is the Gamma function.

In most cases, the higher-order linear fractional PDEs cannot be solved analytically and their solutions cannot be expressed in closed forms, where solutions of such equations are always demand due to physical interests. Anyhow, in the literature, a number of methods have been developed for numerical or analytical solutions for fractional PDEs. The reader is asked to refer to [22–29] in order to know more details about these methods, including their kinds and history, their modifications and conditions for use, their scientific applications, their importance and characteristics, and their relationship including the difference.

The outline of the letter is as follows. In the next section, we utilize some necessary definitions and results from fractional calculus theory. In Section 3, basic idea of the RPS method is represented in order to construct and predict the multiple fractional power series expansion solution. In Section 4, some physical and mathematical linear fractional PDEs of different types and orders are performed in order to illustrate capability and simplicity of the proposed method. The conclusion is given in the final part, Section 5.

2. Overview of fractional calculus theory: mathematical tools and theories

For the concept of fractional derivative we will adopt the Caputo's definition which is a modification of the Riemann–Liouville definition and has the advantage of dealing properly with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case in most physical and engineering processes [21].