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Initial value/boundary value problem for composite fractional relaxation equation

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ABSTRACT

We consider initial value/boundary value problem for composite fractional relaxation equation involving Caputo fractional derivative of order $0<\beta<1$. We prove by means of change of variable that this problem is reduced to initial value/boundary value problem for fractional diffusion equation involving Riemann–Liouville fractional derivative of order $\beta = 1-\alpha$. Then by means of eigenfunctions expansions, we establish the existence and uniqueness of solution.

1. Introduction

Let $d \in \mathbb{N}^*$ and Ω be a bounded open subset of $\mathbb{R}^d$ with boundary $\partial\Omega$ of class $\mathcal{C}^2$. For $T>0$, we set $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$ and we consider the following composite fractional relaxation equation:

$$\begin{cases} \frac{\partial}{\partial t} z(x,t) - \mathcal{L} D_C^{\beta} z(x,t) = f(x,t), \quad (x,t) \in Q, \\ z(x,t) = 0, \quad (x,t) \in \Sigma, \\ z(0) = z^0(x), \quad x \in \Omega, \end{cases} \tag{1}$$

where $0<\beta<1$, the operator $\mathcal{L}$ is given by

$$\mathcal{L}z(x,t) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(A_{ij}(x)\frac{\partial z(x,t)}{\partial x_i}\right) + c(x,t)z(x,t) \tag{2}$$

and is such that there exists a constant $\nu > 0$ such that

$$\nu \sum_{i=1}^{d} \xi_i^2 \leqslant \sum_{i=1}^{d} A_{i,j}(x)\xi_i\xi_j, \quad x \in \overline{\Omega}, \ \xi \in \mathbb{R} \tag{3}$$

and the coefficients satisfy

$$A_{ij} = A_{ji} \in C^1(\overline{\Omega}), \quad c \in C^1(\overline{\Omega}), \ c(x) \leqslant 0, \ x \in \overline{\Omega}. \tag{4}$$

In other words, the operator $\mathcal{L}$ is uniformly elliptic on $\overline{\Omega}$.

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The theory of fractional differential equations has received much attention over the past twenty years, since they are important in describing the natural models such as diffusion processes, stochastic processes, finance and hydrology. We refer for instance to the books [13,10,12,19]. Existence and uniqueness of solutions to fractional evolution equations have been widely studied. For instance, in [21], Zhou et al. discuss the existence and uniqueness of mild solution to a nonlocal Cauchy problem for the fractional evolution equations in an arbitrary Banach space. The existence of mild solutions for abstract fractional differential equations is considered by N'Guérékata et al. in [22]. Some new results for fractional differential inclusions with boundary conditions is given by Nieto et al. in [20]. We also refer to [19] where some results on Cauchy problem for fractional evolution equation involving Caputo derivative are obtained.

Fractional differential equation in (1) with $0 < \beta < 2$ models an oscillation process with fractional damping term. It was formerly treated by Caputo, who provided a preliminary analysis by the Laplace transform. The special cases $\beta = 1/2$ and $\beta = 3/2$ have been discussed by Bagley and Torvik [15]. Existence and qualities properties of semilinear equations associate to (1) have also been studied by Lizama et al. [16,17] by means of (a, k)-regularized resolvent.

On the other hand, fractional diffusion have been widely study. For instance in [1], Oldham and Spanier discuss the relation between a regular diffusion equation and a fractional diffusion equation that contains a first order derivative in space and half order derivative in time. By means of Fourier–Laplace transforms, Mainardi et al. [2,4] expressed the fundamental solutions of Cauchy problems associated to fractional diffusion-wave equations in term of Wright-type functions that can be interpreted as spatial probability density functions evolving in time with similarity properties. Agrawal [5] studied the solutions for a fractional diffusion wave equation defined in a bounded when the fractional time derivative is described in the Caputo sense. Using Laplace transform and finite sine transform technique, the author obtained the general solution in terms of Mittag–Leffler functions. In [18], Yamamoto et al. studied by means of eigenfunctions the initial value/boundary value problems for fractional diffusion equation and apply the results to some inverse problems. Wyss in [6] used Mellin transform theory to obtain a closed form solution of the fractional diffusion equation in terms of Fox's H-function. In [7], Metzler and Klafter used the method of images and the Fourier–Laplace transform technique to solve fractional diffusion equation for different boundary value problems.

In this paper, we prove by means of eigenfunctions expansions that the composite fractional equation (1) has a unique solution. More precisely, we prove by a change of variable that this problem is reduced to initial value/boundary value problem for fractional diffusion equation involving Riemann–Liouville fractional derivative of order $\alpha = 1 - \beta$. Then using the properties of the operator $\mathcal{L}$, we establish by means of eigenfunctions expansion the existence and uniqueness of solution to the fractional diffusion equation and consequently, the existence and uniqueness of solution to the composite fractional equation (1).

Let us notice that the strong motivation of studying the existence of solutions of such equation using eigenfunctions comes from the fact that, the solutions are obtained in Hilbert spaces which are required in some control problems. As far as we know the study of (1) with eigenfunctions expansion is new.

The rest of the paper is organized as follows. In Section 2, we give some basic definitions and results for fractional order integration, Mittag Leffler functions and properties of operator $\mathcal{L}$. In Section 3, we establish the existence and uniqueness of solution to initial value/boundary value problem for fractional diffusion equation involving Riemann–Liouville fractional derivative of order $\alpha = 1 - \beta$ and give some estimations of the solution. Section 4 is devoted to the existence and uniqueness of solution to initial value/boundary value problem for composite fractional relaxation equation (1). Concluding remarks are made in Section 5.

2. Preliminaries

In this section we recall some basic definitions and results on fractional integration and derivative. We also give some properties and existing results on the operator $\mathcal{L}$.

Definition 2.1 ([10,12,11]). Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}_+$ and $\alpha > 0$. Then the expression

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} f(s)ds, \quad t > 0$$

is called the Riemann–Liouville fractional integral of order α of the function f.

The function Γ defined for any $z \in \mathcal{C}$ such that $\mathcal{R}e(z) > 0$ is given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Definition 2.2 ([10,12,11]). Let $f : \mathbb{R}_+ \to \mathbb{R}$. The left Riemann–Liouville fractional derivative of order α of f is defined by

$$D^\alpha_{RL} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)ds, \quad t > 0,$$

where $\alpha \in (n-1, n), \; n \in \mathbb{N}$.